Working through the Distribution: Money in the Short and Long Run*

Guillaume Rocheteau
University of California, Irvine

Pierre-Olivier Weill
University of California, Los Angeles

Tsz-Nga Wong
Bank of Canada

This version: June 2015

Abstract

We construct a tractable model of monetary exchange with search and bargaining that features a non-degenerate distribution of money holdings in which one can study the short-run and long-run effects of changes in the money supply. While money is neutral in the long run, a one-time, money injections in a centralized market with flexible prices and unrestricted participation generates an increase in aggregate real balances and aggregate output in the short run, a decrease in the rate of return of money, and a redistribution of consumption levels across agents. The price level in the short run varies in a non-monotonic fashion with the size of the money injection, e.g., small injections can lead to short-run deflation while large injections always generate inflation. We extend our model to include employment risk and we show that repeated money injections can raise output and welfare when unemployment is high.

*Guillaume Rocheteau: grochete@uci.edu. Pierre-Olivier Weill: poweill@econ.ucla.edu. Tsz-Nga Wong: rwong@bank-banque-canada.ca. We thank for their comments participants at the 2015 West Coast Search-and-Matching Workshop at the Federal Reserve Bank of San Francisco and seminar participants at the Banque de France.

1 Introduction

We develop a tractable model of monetary exchange with random matching and bargaining that features a non-degenerate distribution of money holdings and a non-degenerate distribution of prices. Our model builds on the workhorse model of monetary theory developped by Lagos and Wright (2005) in which agents trade alternatively in decentralized markets, with random search and bargaining, and in centralized markets with competitive pricing. Despite the presence of idiosyncratic risk, the Lagos-Wright model is analytically tractable due to preferences that eliminate wealth effects in order to keep the distribution of money holdings degenerate. Distributional effects have been discarded not because they are thought to be unimportant but because of the view that they make models of monetary economies analytically untractable.[1] Our model provides a simple and natural departure from the Lagos-Wright environment leading to ex-post heterogeneity in money holdings while preserving tractability. By bringing the interplay between risk sharing and self-insurance at the center stage of the analysis, our model generates new insights for classical, yet topical, questions in monetary economics: the short-run and long-run effects of changes in the money supply. Does a one-time increase in the money supply affect aggregate real balances in the short run, thereby creating non-neutralities? Can our model account for the "price puzzle" according to which a contractionary shock to monetary policy generate short-run inflation (Eichenbaum, 1992)? Are the effects of an increase in the money supply monotone with the size of the money injection? Are they long lasting? Can anticipated inflation raise output and welfare in the presence of distributional effects?

In order to answer these questions we adopt a version of the Lagos-Wright model with a single change: we impose a finite (possibly stochastic) bound, $\bar{h}$, on agents' endowment of labor.[2] Whenever the feasibility constraint on agents' labor supply, $h \leq \bar{h}$, binds, wealth effects become operational: individual real balances depend on past trading histories, value functions are strictly concave in money holdings, and the distribution of money holdings is non-degenerate. Yet, the model remains tractable and can be solved in closed form—including distributions and value functions—for a large class of equilibria. When it cannot be solved in closed form, the equilibrium has a simple recursive structure allowing it to be easily computed.

The key ingredients for the tractability of the model are as follows. First, following Rocheteau and Wright (2005) and Lagos and Rocheteau (2005), LRW thereafter, there is heterogeneity in terms of agents' role in pairwise meetings: some agents are always buyers in bilateral matches in the decentralized goods market while other agents are always sellers. This ex-ante heterogeneity captures the idea that agents can have permanent differences in terms of their liquidity needs. Sellers, who are risk-neutral in terms of their centralized-market consumption, do not hold any real balances. As a result, buyers effectively trade with a representative seller. Second, the terms of trade in pairwise meetings are determined by take-it-or-leave-it offers by buyers, which simplifies the fixed-point problem when searching for an equilibrium. Third, the

[1] Recent studies that find that distributional effects of monetary policy are quantitatively important include Doepke and Schneider (2006) and Coibion, Gorodnichenko, Kueng and Silvia (2012).

[2] Lagos and Wright (2005, p.467) do introduce an upper bound on hours of work, $\bar{H}$, but they restrict their attention to equilibria where $H \leq \bar{H}$ does not bind. Such equilibria will be a subset of all the equilibria we are considering.

buyer's disutility of work in the centralized market is linear, which generates a simple policy rule for the accumulation of real balances and out-of-steady-state dynamics that can be easily computed.[3]

We first characterize steady-state equilibria where the distribution of real balances remains constant over time. In such equilibria buyers have a target for their real balances, which represents their desired level of self-insurance. This target increases with their degree of patience and the frequency of trading opportunities in the decentralized goods market. If their labor endowment, $\bar{h}$, is sufficiently large, agents can reach this target in a single period and, as a result, the distribution of real balances across buyers at the beginning of each period is degenerate. This special case is the one the literature has been focusing on. In contrast, if $\bar{h}$ is sufficiently small, then it takes $N \geq 2$ periods, where N is endogenous and depends on preferences and endowments, for an agent to reach his targeted real balances. As a result, the distribution of money holdings is non-degenerate and risk-sharing considerations matter. We will show that provided that the length of a period of time is sufficiently small, ex-post heterogeneity is a generic property of the equilibrium.

For most of the paper we will focus on a class of tractable equilibria where buyers deplete their money holdings in full whenever they are matched with a seller. Under such trading pattern the distribution of real balances is a truncated geometric distribution with $N \in \mathbb{N}$ mass points. This heterogeneity in wealth generates a distribution of consumption levels and prices across matches, with both output levels and prices being higher for richer buyers. Equilibria where buyers do not deplete their money holdings in full in a match are not solvable in closed form, but they can easily be solved numerically due to the recursive structure of the equilibrium.

We then study the transitional dynamics for allocations and prices following a one-time money injection trhough lump-sum transfer to buyers in the centralized market. (We also consider the case of transfers to both buyers and sellers.) If agents can reach their targeted real balances in a single period, $N = 1$, as in the Lagos-Wright model, such a money injection has no real effect since the price level adjusts proportionally to the money supply and the economy returns to its steady state instantly. However, if $N > 1$ then our model features non-trivial transitional dynamics. For tractability, we start the economy at a steady-state equilibrium where the distribution of money holdings has two mass points at the beginning of the period, $N = 2$: there are agents with low real balances, below their target, and agents holding their targeted money holdings. A one-time increase in the money supply raises aggregate real balances, i.e., the price level does not increase as much as the money supply and appears to be sluggish. The economy returns to its steady state in the following centralized market, i.e., the transition only lasts one period.

The reason for why the value of money fails to reach its new steady-state value instantly is because the centralized market cannot reshuffle the units of money among buyers in a way that preserves neutrality. Indeed, in the laissez-faire equilibrium buyers entering the competitive market with no money are constrained by their labor endowment and cannot reach their desired real balances in a single period. Hence, if they receive a lump-sum transfer, they will hold onto it in order to increase their real balances toward their target. If the value of money were to attain its new steady state instantly, then unconstrained buyers would

[3] The model remains tractable under general preferences, as shown in Rocheteau, Weill, and Wong (2015).

accumulate their steady-state target and aggregate real balances would be larger than their steady-state value— a contradiction. Consequently, the anticipated rate of return of money following the money injection is negative, which drives the target for real balances down. The distribution of real balances becomes less disperse in the following decentralized goods market, which raises aggregate output if the seller's production cost is strictly convex and leads to higher welfare by providing risk sharing.

We provide conditions under which the injection of money triggers a deflation in the short run—the value of money rises above its initial-steady-state value—and aggregate output increases. Symmetrically, a contraction of the money supply can generate an increase in the price level in the short run, thereby explaining the "price puzzle" of Eichenbaum (1992). Sufficiently large money injections are always inflationary in both the short and long run, and they make the distribution of money holdings degenerate in the short run. If the initial steady state features a richer heterogeneity, at least three mass points in the distribution of real balances, $N \geq 3$, then the real effects of a one-time money injection are long-lasting.

Finally, we extend the model to incorporate idiosyncratic unemployment risk in addition to the random matching risk in the goods market. Formally, $\bar{h}$ follows a two-state Markov chain where the low state is interpreted as unemployment. We will study a simple class of equilibria where unemployed agents need several periods in order to accumulate their targeted real balances and we will show that the response of the economy to a monetary shock depends on the size of the unemployment rate. We also study repeated money injections leading to a constant money growth rate. An increase in the money growth rate reduces the rate of return of money but it also improves risk-sharing by raising the real balances of the poorest, unemployed agents. Under some conditions on the steady-state unemployment rate and the income of the unemployed the positive risk-sharing effect dominates and social welfare increases.

1.1 Literature review

The earlier search-theoretic models of monetary exchange by Shi (1995) and Trejos and Wright (1995) were kept tractable by restricting money holdings to $\{0, 1\}$. The model was extended by Camera and Corbae (1999) to allow for a more general support for the distribution of money holdings and it was solved numerically by Molico (2006) under the assumption that buyers make take-it-or-leave-it offers to sellers. Zhu (2005) provides an existence result for monetary steady states. Green and Zhou (1998) and Zhou (1999) study a similar environment where goods are indivisible and prices are posted by sellers. They characterize analytically a class of equilibria where all transactions occur at the same price and show that there exist a continuum of such equilibria. In contrast, our model delivers a unique laissez-faire equilibrium. Menzio, Shi, and Sun (2013) consider a related environment with directed search and price posting. They show that the laissez-faire monetary steady state is block-recursive in the sense that policy functions and value functions are independent of the distribution of real balances. This block-recursivity property also holds in our model but it breaks down under money growth implemented with non-proportional transfers (Menzio, Shi, and Sun, 2013, Section 5). Despite this lack of recursivity, we are still able to characterize the equilibrium analytically both at the steady state and out of steady state. In Menzio-Shi-Sun agents' problems are not concave and existence of equilibrium requires the use of monotone comparative statics methods. In contrast, all individual problems in our model are concave, and we can use textbook dynamic programming techniques to establish general properties of value and policy functions. Finally, Sun (2012) extends the quasi-linear environment with competitive search of Rocheteau and Wright (2005) by introducing idiosyncratic shocks on the linear disutility of labor. While the model generates ex-post heterogeneity, the distribution for real balances conditional on the marginal disutility of labor is degenerate.

Closer to what we do, Chiu and Molico (2010, 2011) adopt the alternating-market structure of Lagos and Wright (2005) where some trades take place in competitive markets and others through search and bargaining. They relax the assumption of quasi-linear preferences in order to obtain distributional effects. While the results in Chiu and Molico are numerical, we obtain a tractable model with closed-form solutions. The main two differences between our approach and the one in Chiu and Molico are as follows. First, we assume ex ante heterogeneity between buyers and sellers as in LRW. As a result, the only relevant distribution of money holdings is the one across buyers and this distribution affects the buyer's problem only through its first moment. Second, we adopt the fully linear specification for preferences over the centralized market good, as in LRW, but we add an upper bound on labor supply. This specification implies that buyers supply their full endowment of labor until they reach their targeted real balances. Moreover, it allows us to have the LRW model with a degenerate distribution and linear value function as a particular case. Zhu (2008) constructs a model with alternating market structures and general preferences and achieves tractability by assuming that agents from overlapping generations can trade at most once in the decentralized market with search and bargaining.

This model is related to our earlier work in Rocheteau, Weill, and Wong (2014) with important differences. Rocheteau, Weill, and Wong (2014) describe a competitive economy in continuous time populated with ex-ante identical agents where the idiosyncratic uncertainty takes the form of preferences shocks for lumpy consumption. In contrast, we study a discrete-time economy with search and bargaining and ex-ante heterogenous agents and idiosyncratic uncertainty due to random matching. These ingredients make our model more easily comparable to the New-Monetarist framework. Moreover, the use of discrete time allows us to harness the ex-post heterogeneity, thereby facilitating the study of transitional dynamics, which is a main focus of our paper.

Berentsen, Camera, and Waller (2005) generalize the Lagos-Wright model by assuming two rounds of trade before agents can readjust their money holdings. This assumption generates a non-degenerate distribution of money holdings at the start of the second decentralized market. In contrast to our environment, any money injection in the centralized market is neutral. Moreover, our model generates a rich distribution of money holdings with a single round of pairwise meetings—the distribution can have any number of mass points as well as continuous intervals. Williamson (2006) obtains short-run non-neutralities in the Lagos-Wright model by introducing limited participation while Faig and Li (2009) achieve a similar objective by adopting the Lucas signal extraction problem. In our model all agents can participate in the centralized market in all periods and changes in the money supply are common knowledge.

Wallace (1997) considers an unanticipated change of the money supply in a random matching model with $\{0, 1\}$ money holdings and shows that the short-run effects are predominantly real while the long-run effects are predominantly nominal. Jin and Zhu (2014) generalize the model by assuming a large upper bound on money holdings and by allowing lotteries to set of the terms of trade in order to overcome the indivisibility of money. They show, through numerical examples, that a money injection can have a persistent effect on output and price adjustments are sluggish. Chiu and Molico (2014) study a closely related model with divisible money and no upper bound on money holdings—as in our setting—and show numerically that unanticipated inflation shocks can have persistent effects on output, prices, and welfare.

Finally, Scheinkman and Weiss (1986) in the context of a Bewley economy with competitive markets and aggregate endowment shocks show that one-time money injections generate output and price effects that depend on the state of the economy. Algan, Challe, and Ragot (2011) study temporary and permanent changes in money growth in a Bewley economy with idiosyncratic employment shocks and quasi-linear preferences. They focus on equilibria with two-state wealth distribution that are analogous to our equilibria with $N = 2$. In contrast, our two-sector economy features an idiosyncratic consumption risk due to the assumption that some trades take place under random, pairwise matching and bargaining. In the last part of the paper we combine both random matching risk and unemployment risk in order to investigate how short-run non-neutralities and the optimal inflation rate depend on the unemployment rate.

2 Environment

Time, $t \in \mathbb{N}_0$, is discrete and the horizon infinite. Each period has two stages. In the first stage agents trade in a decentralized market (DM) with pairwise meetings and bargaining. In the second stage they trade in a centralized market (CM). The DM and CM consumption goods are perishable and the CM good is taken as the numéraire.

The economy is populated by two types of agents: a unit measure of *buyers* and a measure n of *sellers*, where these labels refer to an agent's role in the DM. In the first stage buyers want to consume but cannot produce while sellers are able to produce but do not wish to consume. The period-utility function of a buyer is

$$u(y) - h, \tag{1}$$

where $y \in \mathbb{R}_+$ is DM-consumption and h the quantity of labor supplied by the agent in the CM. We allow h to be negative in which case the buyer consumes the numéraire good. We assume that u is strictly concave with $u(0) = 0$, $u'(0) = \infty$, and $u'(+\infty) = 0$. The technology to produce the CM good is linear so that h units of labor generate h units of numéraire. The buyer's endowment of labor is $\bar{h}$. (We will allow $\bar{h}$ to be stochastic in Sections 7 and 8). In contrast to the existing literature we will consider equilibria where the feasibility constraint, $h \leq \bar{h}$, binds for some agents allowing us to depart from a quasi-linear environment with degenerate distributions of money holdings. In order to keep the utility bounded we also impose a lower bound on h, $h \geq \underline{h}$ where $-\underline{h} \geq 0$ can be interpreted as a satiation level for CM consumption. The period-utility function of a seller is

$$-v(y) + c, \tag{2}$$

where $v(y)$ is the disutility of producing y units of the DM good in a pairwise meeting and $c \geq 0$ is the linear utility of consuming the numéraire. The discount factor across periods, $\beta \in (0, 1)$, is common to all agents.

Market structures differ in the DM and CM. In the DM a measure $\alpha \leq \min\{1, n\}$ of bilateral matches composed of one buyer and one seller are formed. The trading mechanism is such that the buyer in a match makes a take-it-or-leave-it offer to the seller. In the CM all agents are price-takers and markets clear.

There exist intertemporal gains from trade when the seller produces y in the DM in exchange for c in the CM with $v(y) \leq c \leq u(y)$. However, these gains from trade cannot be exploited with unsecured credit since buyers cannot commit to repay their debt and there is no monitoring. There is an intrinsically useless, perfectly divisible and storable asset called money that agents can (but don't have to) use as a medium of exchange to overcome these frictions. We use M_t to denote the money supply in the DM of period t. The price of money in terms of the numéraire is ϕ_t. The gross rate of return of money is denoted $R_t \equiv \phi_t/\phi_{t-1}$.

Full insurance

Suppose that buyers can commit to an insurance contract according to which they supply h units of labor every period in exchange for a consumption level y in the (observable) event they are matched in the DM. The expected utility of the buyer in each period is $\alpha u(y) - h$. The total CM output produced by buyers, h, is promised to the α sellers who are matched in the DM, $c = h/\alpha$. Sellers are willing to go along with this allocation if $c \geq v(y)$, i.e., their consumption is greater than their disutility of production. Hence, the insurance contract among buyers, (h, y), solves the following maximization problem:

$$\max_{y,c,h \leq \bar{h}} [\alpha u(y) - h] \quad \text{s.t.} \quad c = \frac{h}{\alpha} \geq v(y).$$

The solution is $y = y^*$, where $u'(y^*) = v'(y^*)$, if $\bar{h} \geq \alpha v(y^*)$. Otherwise, $y = v^{-1}\left(\bar{h}/\alpha\right) < y^*$. So provided that labor endowments are sufficiently large, the full-insurance allocation equalizes the marginal utility of consumption of the buyer and the marginal disutility of the seller, as in LRW. In contrast, if the labor endowment is not large enough to implement y^* then the full-insurance allocation is such that DM output is maximum, $h = \bar{h}$, and $u'(y) > v'(y)$.

3 Equilibrium

We characterize an equilibrium in three steps. First, we study the decision problem of a buyer by taking as given the sequence of rates of return for currency, $\{R_t\}_{t=1}^{+\infty}$. Second, given the buyer's optimal consumption/saving decisions we write down the law of motion for the distribution of real balances. Third, we clear the money market in every CM in order to obtain the value of money, $\{\phi_t\}_{t=0}^{+\infty}$, and hence its rate of return.

Value functions Consider first the problem of a buyer at the beginning of the CM of period t holding z real balances (money balances expressed in terms of the period-t CM good). We assume the following conditions hold:

$$\beta R_t < 1 \quad \text{and} \quad \lim_{t\to\infty} \beta^t \prod_{i=1}^{t} R_i = 0. \tag{3}$$

According to (3) the rate of return of money is less than the rate of time preference, β^{-1}, which guarantees that sellers have no incentive to hold money from one period to the next, and the present value from hoarding a unit of real balances goes to 0 as time goes to infinity. The value function of a buyer solves:

$$W_t(z) = \max_{h,z'} \{-h + \beta V_{t+1}(z')\} \tag{4}$$

$$\text{s.t.} \quad z' = R_{t+1}(h + z) \tag{5}$$

$$z' \geq 0, \quad h \in \left[\underline{h}, \bar{h}\right]. \tag{6}$$

According to (4) the buyer chooses his supply of labor, h, and next-period real balances, z', in order to maximize his discounted continuation value in $t+1$ net of the disutility of work. The budget identity, (5), specifies that the next-period real balances are equal to the sum of the current wealth and labor income multiplied by the gross rate of return of money. From (6), the buyer's problem is subject to a non-negativity constraint for real balances and a feasibility constraint on the labor supply. The value functions are indexed by t as the gross rate of return of money, R_t, might vary over time.

In the DM each matched buyer makes a take-it-or-leave-it offer, (y, p), to a seller, where y is the DM output to be produced by the seller and p is the payment in terms of real balances by the buyer.[4] This payment must satisfy the feasibility constraint $p \leq z$ (since buyers' IOUs are not accepted by sellers due to lack of commitment and lack of monitoring). It must also satisfy the individual rationality constraint of the seller according to which the payment must be at least equal to the disutility of production, $-v(y) + p \geq 0$. Indeed, sellers get no surplus in the DM and hence, from (3), they have no motive for carrying real balances from one period to the next. Consequently, sellers spend all the money they accumulate in the DM in the following CM.[5] The seller's participation constraint will hold at equality, $v(y) = p$, as otherwise the buyer would have an incentive to reduce the size of the payment for the same output level. Hence, the lifetime expected discounted utility of a buyer at the beginning of the DM is:

$$V_t(z) = \alpha \max_{p\leq z} \left[\omega(p) + W_t\left(z - p\right)\right] + (1-\alpha) W_t(z), \tag{7}$$

where $\omega(p) \equiv u \circ v^{-1}(p)$. With probability α the buyer is matched in the DM in which case he chooses an output level, y, in exchange for $p = v(y)$ units of real balances. With probability $1 - \alpha$ the buyer is unmatched and enters the next CM with z real balances.

[4] The model remains tractable under competitive pricing. See Rocheteau, Weill and, Wong (2014) for a related model in continuous-time where households trade in a competitive market.

[5] This is the place where the assumption of ex-ante heterogenous buyers and sellers borrowed from Rocheteau and Wright (2005) and Lagos and Rocheteau (2005) plays a crucial role to keep the model tractable. If instead agents could be both buyers and sellers in the DM, as in Lagos and Wright (2005), then the outcome of the bargaining would depend on both the money holdings of the buyer and the money holdings of the seller in the match and the distribution of money holdings would be a state variable in the agent's problem. For such a model, see Chiu and Molico (2011).

We now prove the existence, continuity, and differentiability of the value functions, $W_t(z)$. From (4)-(6) and (7) we define W_t recursively as follows:

$$W_t(z) = \max_{p,z'} \left\{ z - \frac{z'}{R_{t+1}} + \beta\alpha \left[\omega(p) + W_{t+1}(z'-p)\right] + \beta(1-\alpha)W_{t+1}(z') \right\}$$
$$\text{s.t.} \quad z' \in \left[R_{t+1}\left(z+\underline{h}\right), R_{t+1}\left(z+\bar{h}\right)\right] \quad \text{and } p \leq z'. \tag{8}$$

Lemma 1 *There exists a unique W_t solution to (8) in the space of continuous and bounded functions. Moreover, W_t is concave and differentiable.*

In order to prove Lemma 1 we use that (8) defines a contraction mapping from the set of bounded functions defined over $\mathbb{N} \times \mathbb{R}_+$ into iself. As a result the fixed point of this functional equation is continuous and bounded. Moreover, concavity is preserved by this mapping according to Theorem 4.7 in Stokey and Lucas (1989). The differentiability of W_t is less obvious because the choice for (p, z') is not necessarily in the interior of the feasible domain. Hence, we apply results from Rincón-Zapatero and Santos (2009) that holds under condition (3).

Choice of real balances In the following we focus on equilibria such that the constraint, $h \geq \underline{h}$, never binds. Let $\xi_t(z)$ denote the Lagrange multiplier associated with $h \leq \bar{h}$. Substituting $h = z'/R_{t+1} - z$ from (5) into the objective we can rewrite the buyer's problem as:

$$W_t(z) = z + R_{t+1}^{-1} \max_{z' \geq 0} \left\{ -z' + \beta R_{t+1} V_{t+1}(z') + \xi_t \left[R_{t+1}(\bar{h}+z) - z'\right] \right\}. \tag{9}$$

If $\xi_t = 0$, then the second term on the right side of (9) is independent of z, the choice of next-period real balances is independent of current wealth, and W_t is linear, as in Lagos and Rocheteau (2005, Eq. (9)). However, if the feasibility constraint on labor binds, $\xi_t > 0$, then the choice of real balances is no longer independent of current wealth and W_t is no longer linear—the two key ingredients of the tractability of the Lagos-Wright model. The envelope theorem applied to (9) gives:

$$W_t'(z) \equiv \lambda_t(z) = 1 + \xi_t. \tag{10}$$

The first-order condition for the choice of real balances is

$$-\lambda_t(z) + R_{t+1}\beta V_{t+1}'(z') \leq 0, \quad \text{“} = \text{” if } z' > 0. \tag{11}$$

The cost of accumulating real balances is measured by $\lambda_t = 1 + \xi_t$. The marginal benefit of real balances in the CM of period t is the discounted marginal value of real balances in the DM of $t+1$ times the gross rate of return of real balances, $R_{t+1}\beta V_{t+1}'(z')$. We define the buyer's targeted real balances for $t+1$, z_{t+1}^*, as the solution to (11) when $\xi_t = 0$, i.e.,

$$R_{t+1}\beta V_{t+1}'(z_{t+1}^*) = 1. \tag{12}$$

The target is the buyer's choice when $h \leq \bar{h}$ does not bind, $z + \bar{h} \geq z_{t+1}^*/R_{t+1}$. It equalizes the marginal disutility of labor, one, with the discounted marginal value of real balances in the next DM.

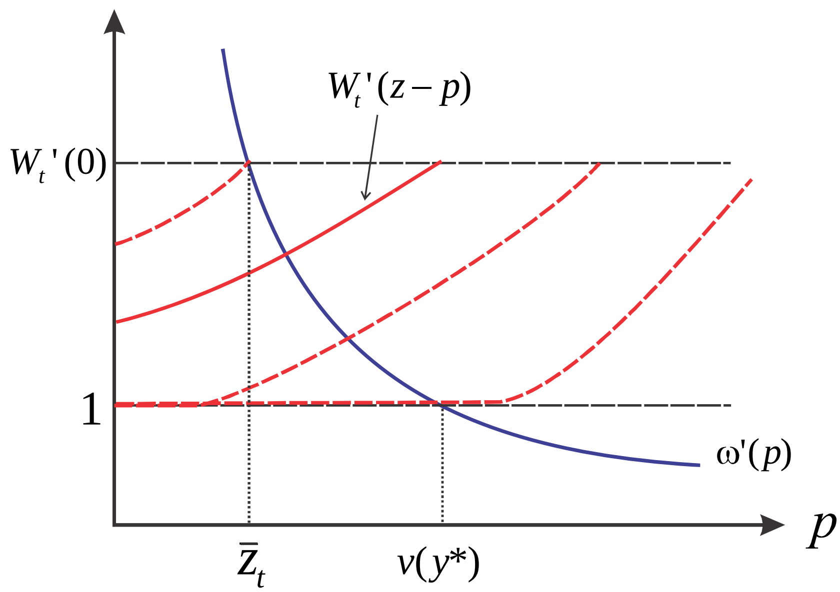

Figure 1: Bargaining outcome

Terms of trade in the DM. The solution to the maximization problem on the right side of (7) is $p = v(y)$ and

$$\omega'(p) = W_t'(z-p) \text{ if } \omega'(z) < W_t'(0) \quad (13)$$

$$p = z \text{ otherwise.} \quad (14)$$

According to (13) the buyer equalizes his marginal utility from spending a unit of real balances in the DM, $u'(y)/v'(y)$, with the marginal value of real balances in the CM as measured by W_t'. We represent (13) in Figure 1 where the left side is the blue downward-sloping curve and the right side is the red upward-sloping curve. From the concavity of W it follows that $W_t'(z-p)$ is non-increasing in z. Hence, as z increases the red upward-sloping curve moves downward as illustrated by the two dashed curves located underneath the plain upward-sloping curve. Those curves are horizontal for low values of p because buyers enter the next CM with enough real balances to reach their targeted real balances, in which case the marginal utility of real balances is equal to one. It follows that payment, p, and output, y, are non-decreasing in the buyer's real balances. Similarly, if we denote post-trade real balances by $\tilde{z} = z - p$ then $\omega'(z-\tilde{z}) = W_t'(\tilde{z})$. Hence, post-trade real balances are weakly increasing with pre-trade real balances.

From (14) if the marginal utility from spending real balances in the DM is larger than the marginal value of money in the CM when money holdings are depleted in full, $\omega'(z) \geq W_t'(0)$, then the buyer spends all his real balances. We denote by $\bar{z}_t$ the threshold below which there is full depletion of real balances. It solves

$$\omega'(\bar{z}_t) = W_t'(0). \quad (15)$$

If $W_t'(0) < +\infty$, which follows from the fact that a buyer can consume at least $v(R_{t+1}\bar{h}) > 0$ in the next DM, then $\bar{z}_t > 0$. In Figure 1 we represent $W'(\bar{z}_t - p)$ by a dashed curve located above the plain upward-sloping curve. It intersects the horizontal line given by $W_t'(0)$ and $\omega'(p)$ when $p = \bar{z}_t$.

There is another threshold for real balances, $\hat{z}_t$, above which buyers are unconstrained by their labor endowment in the following CM given their post-trade real balances, in which case $W_t'[\hat{z}_t - p(\hat{z}_t)] = 1$. From (13) it follows that for all $z \geq \hat{z}_t$ (assuming $h_t \geq \underline{h}$ does not bind), $y = y^*$ and $p = v(y^*)$. Hence, the threshold $\hat{z}_t$ is

$$\hat{z}_t = v(y^*) + \frac{z_{t+1}^*}{R_{t+1}} - \bar{h}. \tag{16}$$

In order to be unconstrained in the following CM the buyer must hold enough real balances so that he can pay for the first-best level of output in the DM, which requires $v(y^*)$, and he can supplement his labor endowment to finance his next-period targeted real balances, $z_{t+1}^*/R_{t+1} - \bar{h}$.

If $\bar{h} < z_{t+1}^*/R_{t+1}$ then a buyer with depleted real balances cannot reach his target so that $W_t'(0) > 1$ and $\bar{z}_t < \hat{z}_t$. So in contrast to the bargaining outcome in the LRW model there is an interval of real balances, $(\bar{z}_t, \hat{z}_t)$, for which buyers spend a fraction of their real balances even though they consume less than y^*. Buyers find it optimal not to spend all their real balances because they look forward at the next CM and anticipate that they will not be able to reach their targeted real balances. We summarize the outcome of the bargaining problem in the following proposition and Figure 2.

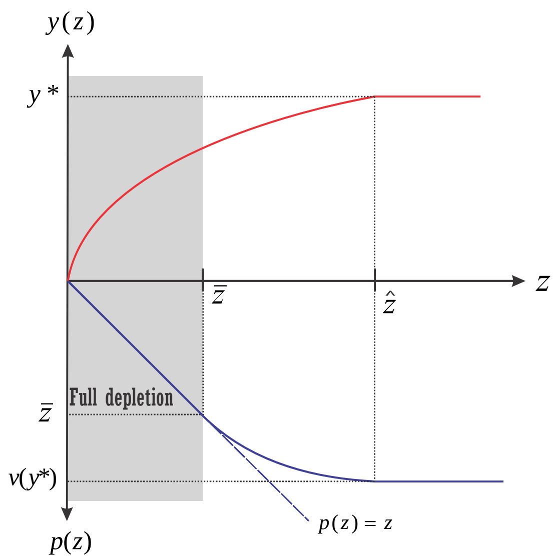

Figure 2: Terms of trade in DM pairwise meetings

Proposition 1 ***(Bargaining outcome)*** *Assume $W_t'(0) > 1$, i.e., $\bar{h} < z_{t+1}^*/R_{t+1}$. There are two thresholds for real balances, $\bar{z}_t < \hat{z}_t$ defined in (15) and (16), such that the outcome of the buyer's take-it-or-leave-it bargaining game satisfies:*

1. *For all $z \leq \bar{z}_t$, $p = z$ and $y = v^{-1}(z)$.*

2. *For all* $z \in (\bar{z}_t, \hat{z}_t)$, $p < z$, *and* $y = v^{-1}(p)$ *are increasing with real balances.*

3. *For all* $z \geq \hat{z}_t$, $p = v(y^*)$, *and* $y = y^*$.

The marginal value of real balances at the beginning of the DM is

$$V_t'(z) = \alpha\omega'\left[p_t(z)\right] + (1-\alpha)\lambda_t(z), \tag{17}$$

where $p_t(z)$ is the solution to the bargaining problem, (13)-(14). The marginal value of real balances in the DM is equal to the marginal utility of DM consumption with probability α (a match occurs) and the marginal utility of real balances in the following CM with probability $1-\alpha$ (the buyer is unmatched). We substitute $V_{t+1}'(z')$ by its expression given by (17) into (11) to obtain the law of motion for the marginal value of real balances:

$$\lambda_t(z) = R_{t+1}\beta\left\{\alpha\omega'\left[p_{t+1}(z')\right] + (1-\alpha)\lambda_{t+1}(z')\right\}. \tag{18}$$

The marginal value of real balances at time t is equal to the discounted marginal value of R_{t+1} real balances in $t+1$ plus the present value of a liquidity term equal to $\alpha\left\{\omega'\left[p_{t+1}(z')\right] - \lambda_{t+1}(z')\right\}$ which corresponds to the buyer's expected marginal surplus from spending a unit of real balances in a DM match.

Distribution of real balances. Let us turn to the law of motion for the distribution of real balances. We denote $F_t(z)$ the distribution of real balances at the beginning of period t. The distribution at time $t+1$ is given by:

$$F_{t+1}(z) = \int \alpha\mathbb{I}_{\{z_{t+1}[x-p_t(x)]\leq z\}} + (1-\alpha)\mathbb{I}_{\{z_{t+1}(x)\leq z\}} dF_t(x), \tag{19}$$

where $z_{t+1}(x)$ is the policy function derived from (13)-(14) that specifies the choice of real balances in $t+1$ given the real balances at the beginning of the CM in t. It is given by:

$$z_{t+1}(x) = \min\{(x+\bar{h})R_{t+1}, z_{t+1}^*\}.$$

Either the buyer can reach his target, z_{t+1}^*, or he supplies all his labor so that his total wealth is composed of his initial wealth and his labor endowment, $x+\bar{h}$. This wealth is capitalized according to the gross real rate of return of fiat money. The first term underneath the integral on the right side of (19) represents the measure α of buyers who are matched in the DM and enter the CM with $x - p_t(x)$ real balances, where x is their pre-trade real balances. In the CM they accumulate $z_{t+1}\left[x - p_t(x)\right]$ for period $t+1$. The second term represents the unmatched buyers.

Value of money Finally, the value of money is determined by the following money market clearing condition:

$$\phi_t M_t = \int x dF_t(x). \tag{20}$$

Hence, the rate of return of money is

$$R_{t+1} = \frac{\phi_{t+1}}{\phi_t} = \frac{M_t}{M_{t+1}}\frac{\int x dF_{t+1}(x)}{\int x dF_t(x)}. \tag{21}$$

Definition 1 *Given some initial distribution,* F_0*, an equilibrium is a sequence,* $\{F_t, \phi_t, R_{t+1}\}_{t=0}^{+\infty}$*, that solves (19), (20), and (21).*

A steady-state equilibrium is such that $\{F_t, \phi_t\}$ is constant over time and the gross rate of return of money is $R_t = \phi_{t+1}/\phi_t = 1$. From (18) with $\xi(z^*) = 0$ the targeted real balances solve:

$$\frac{u'[y(z^*)]}{v'[y(z^*)]} = 1 + \frac{r}{\alpha}, \tag{22}$$

where from (13)-(14), $y(z^*)$ is the solution to

$$\begin{aligned} W'[z^* - v(y^*)] &= 1 + \frac{r}{\alpha} \text{ if } \omega'(z^*) < W'(0) \\ z^* &= v(y^*) \text{ otherwise.} \end{aligned}$$

At the targeted real balances the ratio of the marginal utility of DM consumption to the marginal disutility of DM production is equal to the marginal disutility of labor in the CM, one, plus the average holding cost of real balances. This cost is equal to the rate of time preference multiplied by the average period length until a match in the DM occurs, $1/\alpha$.

Given z^* we can define the transition probabilities, $Q(z, z')$, as the probability that a buyer with $z \in [0, z^*]$ real balances at the beginning of period t ends up with z' in the following period, $t+1$. They satisfy:

$$Q(z, z') = \left\{ \begin{array}{c} \alpha \\ 1 - \alpha \end{array} \right. \text{ if } z' \left\{ \begin{array}{l} = \min\{z - p(z) + \bar{h}, z^*\} \\ = \min\{z + \bar{h}, z^*\} \end{array} \right. . \tag{23}$$

From the continuity of $p(z)$ it follows that Q satisfies the Feller property so that there exists a steady-state distribution of real balances associated with Q (Theorem, 12.10, Stokey and Lucas, 1989). Moreover, Q satisfies Assumption 12.1 in Stokey and Lucas (1989) so that F is unique. It follows that the steady-state monetary equilibrium with a constant money supply exists and is unique.

4 Money in the long run

We focus on equilibria where it takes $N > 1$ consecutive rounds of CM trades for a buyer with depleted money holdings to rebuild his targeted real balances if he remains unmatched in all DMs. (The case $N = 1$ is the LRW model.) Moreover, we will be focusing on equilibria where buyers deplete all their real balances in the DM, $\bar{z} < z^*$.

Targeted real balances From (22) the buyer's targeted real balances, $z^* \in \left((N-1)\bar{h}, N\bar{h}\right]$, solves

$$\omega'(z^*) = 1 + \frac{r}{\alpha}. \tag{24}$$

As buyers become more patient, or as the frequency of matches increases, the targeted real balances increase. The condition $z^* \in \left((N-1)\bar{h}, N\bar{h}\right]$ can be reexpressed as

$$(N-1)\bar{h} < \omega'^{-1}\left(1 + \frac{r}{\alpha}\right) \leq N\bar{h}. \tag{25}$$

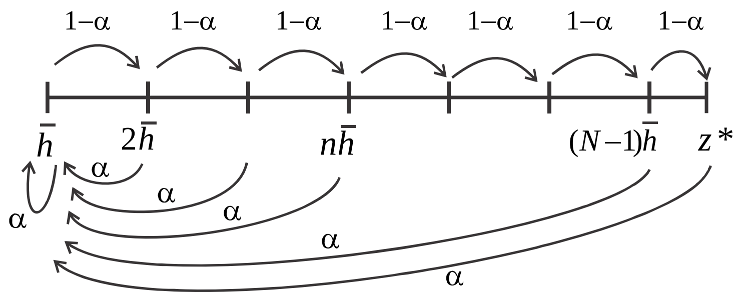

Figure 3: Support of the distribution of real balances

Distribution of real balances The support of the distribution of real balances across buyers at the beginning of a period is $\{\bar{h}, 2\bar{h}, ..., (N-1)\bar{h}, z^*\}$. Indeed, buyers increase their real balances by the size of their labor endowment, $\bar{h}$, until they reach their target. See Figure 3. The distribution F is composed of N mass points, $\{\mu_n\}_{n=1}^N$, where μ_n is the measure of buyers holdings $n\bar{h}$ for all $n \in \{1, ..., N-1\}$ and μ_N is the measure of buyers holding their target, z^*. We have:

$$\mu_1 = \alpha \tag{26}$$

$$\mu_n = (1-\alpha)\mu_{n-1} \text{ for all } n \in \{2, N-1\} \tag{27}$$

$$\alpha\mu_N = (1-\alpha)\mu_{N-1}. \tag{28}$$

According to (26) each buyer is matched with a seller with probability α, in which case he spends all his real balances (since we are focusing on equilibria with full depletion). By the Law of Large Numbers the measure of buyers entering the CM with depleted money balances is α. Those buyers supply their full endowment of labor in order to start the following period with $z_1 = \bar{h}$ real balances. According to (27) the measure of agents holding $z_n = n\bar{h} \in (z_1, z^*)$ is equal to the measure of buyers holding z_{n-1}, μ_{n-1}, times the probability that they were unmatched in the last DM round, $1-\alpha$, so that such buyers add $\bar{h}$ to their existing real balances. Finally, the measure of agents holding the targeted real balances is determined such that the flow of buyers with the targeted real balances who are matched in the DM, $\alpha\mu_N$, is equal to the flow of buyers holding z_{N-1} who are unmatched in the DM and reach z^* in the next CM. It is easy from (26)-(28) to solve for the distribution of real balances in closed form:

$$\mu_n = \alpha(1-\alpha)^{n-1} \text{ for all } n = 1, ..., N-1 \tag{29}$$

$$\mu_N = (1-\alpha)^{N-1}. \tag{30}$$

From (29)-(30) the distribution of real balances is a truncated geometric distribution.[6]

The model features a distribution of nominal prices in the DM. The unit price of the DM output for a buyer holding $z_n = n\bar{h}$ real balances is $z_n/v^{-1}(z_n)\phi$, which is increasing in z_n if v is strictly convex. So the

[6]Green and Zhou (1998), Zhou (1999), and Rocheteau (2000) also find geometric distributions of money holdings in search model with price posting and indivisible goods. However, the dynamics of individual real balances are different as individual accumulate and deplete real balances one unit at a time.

richest agents in the DM purchase larger quantities and pay a higher price to compensate sellers for their convex disutility of production. The fraction of the transactions taking place at that price is μ_n.

Value of money and prices. Aggregate real balances are $\phi M = \sum_{n=1}^{N} \mu_n z_n$. From (29)-(30), and after some calculation, this gives

$$\phi M = \bar{h}\frac{\left\{1-(1-\alpha)^{N-1}\left[(N-1)\alpha+1\right]\right\}}{\alpha} + (1-\alpha)^{N-1}z^*. \tag{31}$$

Aggregate real balances do no depend on the nominal money supply and hence money is neutral in the long run. For a given N the value of money increases with the buyer's labor endowment, $\bar{h}$, and it decreases with the rate of time preference, r.

Marginal value of real balances Next, we determine the marginal value of real balances, $\lambda(z) = 1+\xi(z)$, recursively. Suppose $z \in (z^* - \bar{h}, z^*)$. If the buyer can reach his targeted real balances by supplying less than $\bar{h}$, then the feasibility constraint on labor is slack, $\xi(z) = 0$. As a result, $\lambda(z) = 1$ and $W(z)$ is linear. From (18),

$$\lambda(z) = \beta\left[\alpha\omega'\left(z+\bar{h}\right) + (1-\alpha)\lambda(z+\bar{h})\right], \quad \text{for all } z \leq z^* - \bar{h}. \tag{32}$$

If a buyer enters the CM with $z \leq z^* - \bar{h}$ real balances then he supplies his endowment of labor and enters the next period with $z + \bar{h}$. With probability α the buyer is matched and spends all his real balances. The marginal value of a unit of money is then $\omega'\left(z+\bar{h}\right)$. With probability $1-\alpha$ the buyer is unmatched and enters the CM with $z + \bar{h}$, in which case the marginal value of money is $\lambda(z+\bar{h})$. The difference equation (32) can be solved in closed form to give:

$$\lambda(z) = 1 + \alpha\sum_{j=1}^{+\infty}\beta^j(1-\alpha)^{j-1}\left[\omega'\left(z+j\bar{h}\right) - \omega'(z^*)\right]^+, \tag{33}$$

where $[x]^+ = \max\{x, 0\}$. The marginal value of money is equal to one, the marginal disutility of work, plus the discounted sum of the differences between the marginal utility of lumpy consumption given the buyer's real balances at a point in time and his marginal utility of consumption at the targeted real balances. It is easy to check that $\lambda(z) = W'(z)$ is decreasing in z (from the concavity of $u \circ v^{-1}(z)$) and continuous.

Given $\lambda(z)$ we can obtain the value function, $W(z)$, in closed form. At his targeted real balances the lifetime expected utility of a buyer is

$$W(z^*) = \beta\left\{\alpha\left[\omega(z^*) + W(0)\right] + (1-\alpha)W(z^*)\right\}. \tag{34}$$

The buyer does not need to readjust his real balances, and hence he incurs no cost in the CM. In the following DM he is matched with probability α in which case he depletes all his money balances. If he is unmatched he enters the subsequent CM with his targeted real balances. Multiplying both sides of (34) by β^{-1} and using that $W(z^*) - W(0) = \int_0^{z^*}\lambda(x)dx$, $W(z^*)$ can be rewritten as

$$rW(z^*) = \alpha\left[\omega(z^*) - \int_0^{z^*}\lambda(x)dx\right]. \tag{35}$$

Given $W(z^*)$ we obtain $W(z)$ as follows:

$$W(z) = W(z^*) - \int_z^{z^*} \lambda(x)dx = \frac{\alpha}{r}\left[\omega(z^*) - \int_0^{z^*} \lambda(x)dx\right] - \int_z^{z^*} \lambda(x)dx. \tag{36}$$

We represent $W(z)$ in Figure 4.

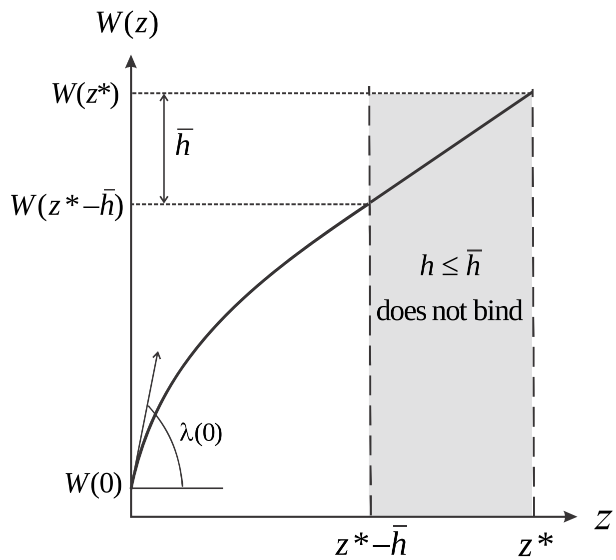

Figure 4: Value function at a steady-state monetary equilibrium

The condition for full depletion of real balances is $\omega'(z^*) \geq \lambda(0)$. The marginal utility that the buyer gets from spending his last unit of real balances, $\omega'(z^*)$, must be greater than the marginal utility from holding onto this unit of money, $\lambda(0)$. From (33) the condition for full depletion is

$$\omega'(z^*) - 1 = \frac{r}{\alpha} \geq \alpha \sum_{j=1}^{+\infty} \beta^j (1-\alpha)^{j-1} \left[\omega'\left(j\bar{h}\right) - \omega'(z^*)\right]^+, \tag{37}$$

We represent the condition (37) by a grey area in Figure 5. The dotted lines represent the conditions in (25). The case studied in LRW, $N = 1$, requires the endowment in labor, $\bar{h}$, to be large so that the buyer can readjust his money balances in a single period. If the endowment is such that $\omega'(\bar{h}) > 1 + r/\alpha$ then it will take more than one period for the buyer to reach his targeted real balances. In the Appendix A2 we provide numerical examples to illustrate how N varies with fundamentals, α, $\bar{h}$, and r.

We can now define a steady state equilibrium as follows.

Definition 2 *A steady-state, monetary equilibrium with full depletion of real balances is a list,* $(N, z^*, \phi, \{\mu_n\}_{n=1}^N)$, *that solves (22), (25), (29)-(30), (31), and (37).*

Provided that the condition for full depletion, (37), holds one can construct a steady-state equilibrium as follows. From (22) one determines the targeted real balances, z^*. We use (25) to compute the number of periods it takes to reach the target, N. Given N and z^* the steady-state distribution of real balances is obtained from (29)-(30). Finally, the value of money is obtained from (31).

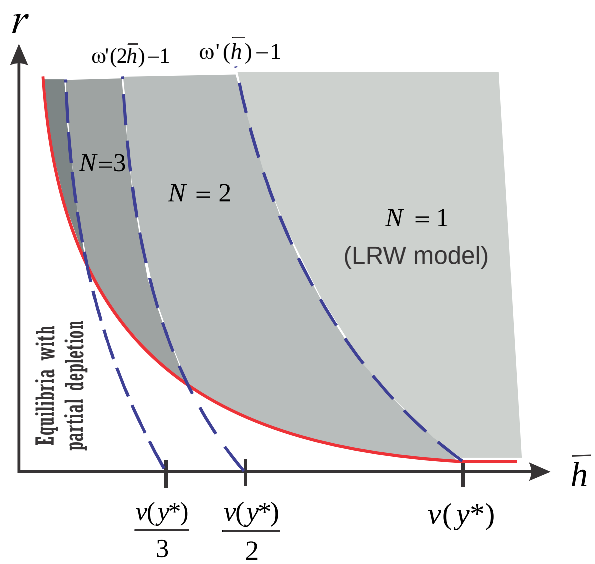

Figure 5: Existence of equilibria with full depletion of real balances

Proposition 2 ***(Existence of steady-state monetary equilibria with full depletion.)*** *If (37) holds, then there exists a steady-state monetary equilibrium with full depletion.*

Provided that the buyer's labor endowment is not too low and agents are not too patient, there exists an equilibrium where buyers deplete their money holdings in full whenever they are matched in the DM.

Finally, we characterize equilibria with full depletion when the length of a period of time, denoted Δ, is small. Such limits are relevant because search-theoretic models of monetary exchange are typically viewed as models of high-frequency trade. The variables that depend on the length of a period of time are r, α, and $\bar{h}$. We index these variables by Δ and we denote $r_\Delta = r\Delta$, $\alpha_\Delta = \alpha\Delta$, and $\bar{h}_\Delta = \bar{h}\Delta$. From (24) the targeted real balances, z^*, do not depend on Δ.

Proposition 3 ***(Continuous-time limit.)*** *If $\Delta < \omega'^{-1}\left(1+\frac{r}{\alpha}\right)/\bar{h}$ then any equilibrium features a non-degenerate distribution of real balances. Consider a sequence of equilibria with full depletion of real balances as $\Delta \to 0$. Then, $F(z) \to 1 - e^{-\frac{\alpha z}{\bar{h}}}$ and $\phi M \to \bar{h}\left(1 - e^{-\alpha T}\right)/\alpha$ where $T = z^*/\bar{h}$. At the limit, the condition for full depletion, (37), becomes*

$$\frac{r}{\alpha} \geq \int_0^T \alpha e^{-(r+\alpha)t}\left[\omega'\left(t\bar{h}\right) - \omega'(z^*)\right] dt. \tag{38}$$

At sufficiently high frequency, the equilibrium features a non-degenerate distribution of real balances. In order to determine the limit of the distribution we denote $t = n\Delta < T$ and $z_t = t\bar{h} < z^*$ and we take the limit as Δ goes to 0 and $n\Delta$ is kept constant and equal to t. When the length of a time interval becomes very small, the distribution of real balances converges to a truncated geometric distribution. It can be checked that these limits are analogous to the ones of the continuous-time economy of Rocheteau, Weill, and Wong (2015).

A numerical example with partial depletion of real balances

Equilibria that feature partial depletion of real balances cannot be solved in closed form. However, the model is still very tractable numerically as the equilibrium can be solved recursively. First, $W(z)$ is the fixed point of a contraction mapping, (8), that is independent of the distribution of real balances. Second, once $W(z)$ and the associated policy function, $p(z)$, are obtained (by iterations of the Bellman equation) we can use them to compute the distribution of real balances. Indeed, the cumulative distribution of real balances is the solution to the following functional equation:

$$(1-\alpha)\left\{F(z)-F\left[(z-\bar{h})^{+}\right]\right\}=\alpha\left\{F\left[g(z)\right]-F(z^{+})\right\}, \tag{39}$$

for all $z \geq \bar{h}$, where $g(z)$ is defined implicitly by

$$g-p(g)=z-\bar{h}.$$

By convention, $g(\bar{h})=\bar{z}$. We interpret g as the required real balances to leave a DM match with exactly $z-\bar{h}$. The left side of (39) corresponds to the measure of buyers with real balances no greater than z but strictly larger than $z-\bar{h}$ who are unmatched in the DM. Those buyers enter the following period with more than z real balances. The right side of (39) corresponds to buyers who hold more than z real balances and who have enough to leave a match with at least z, and such buyers are matched in the DM. These buyers enter the follow period with no more than z. In a steady state, the two measures of buyers given by the left and right sides of (39) must be equal. It should be clear from (39) that once we know the policy function $p(z)$ we should be able to compute $F(z)$. The distribution is computed numerically as follows. We generate a large number of long trading histories and we use the resulting terminal real balances to compute the empirical distribution of real balances. (The algorithm is detailed in the appendix.)

In the following we show a numerical example obtained with the following parameter values: $\bar{h}=0.5$, $\alpha=0.8$, $u(y)=3y^{1/3}$, $r=0.04$. The left panel of Figure 6 plots the distribution of real balances. The middle panel plots the value function, and the right panel plots post-trade real balances.

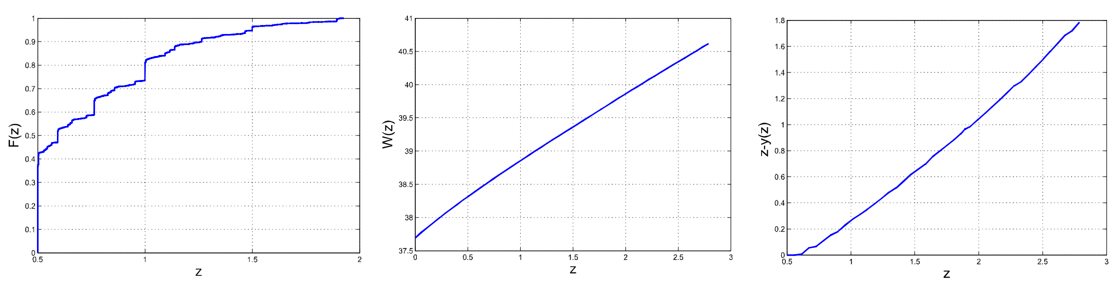

Figure 6: Numerical example of an equilibrium with partial depletion

The left panel shows that there is a mass of buyers, about 40%, holding real balances equal to their endowment, $\bar{h}=0.5$. This mass is less than the frequency of a trading opportunity in the DM, $\alpha=0.8$, because the equilibrium features partial depletion, i.e., not all buyers who are matched in the DM deplete their money holdings in full. According to the right panel the threshold for full depletion is about $\bar{z}_1 \approx 0.6$ which is less than the target, $z^* \approx 2.7$. Moreover, it takes $N = 6$ periods for a buyer with depleted money holdings to accumulate the targeted real balances. One can also see that the distribution, F, has multiple mass points. For instance, there are about 10 percent of buyers holding $2\bar{h} = 1$. Buyers with 1 unit of real balances leave a DM match with $z - y(z) \approx 0.25$ and then they accumulate $\bar{h}$ in the following CM. As a result there is also a mass of buyers at $z \approx 0.75$.

5 Money in the short run

We now study the short-run effects of a money injection. Following LRW, we assume that an agent's type as buyer or seller is observable so that the monetary authority can transfer $(\gamma - 1)M$, with $\gamma > 1$, in a lump-sum fashion to all buyers at the time they enter the CM of $t = 0$.[7] The change in the money supply is common knowledge among all agents.

As a benchmark, consider first equilibria with $N = 1$, i.e., the distribution of money holdings is degenerate at the beginning of each period. Such equilibria exist if $\omega'\left(\bar{h}\right) < 1+r/\alpha$. It is easy to check that following the one-time money injection there is an equilibrium with $\phi_t = \phi_0 = z^*/\gamma M$ for all $t \geq 1$ where $\omega'\left(z^*\right) = 1+r/\alpha$. Indeed, at the beginning of the CM of $t = 0$ there is measure $1 - \alpha$ of buyers holding M and a measure α holding 0. Following the transfer, the former hold γM while the latter hold $(\gamma - 1)M$. If the rate of return of money is $R_1 = 1$, both types want to hold z^*, which is feasible since $\bar{h} > z^*$. Consequently, the value of money adjusts instantly to its new steady-state value and the money injection has no real effect.

Proposition 4 ***(Neutral money injections)*** *Suppose the economy is initially at a steady state with $N = 1$. A one-time money injection, $(\gamma-1)M$, in the CM of $t = 0$ leaves aggregate real balances, $\phi_t\gamma M$, unaffected. The value of money adjusts instantly to its new steady-state value and $R_t = 1$ for all $t \geq 1$.*

In the rest of this section we focus on equilibria with $N = 2$ as these equilibria can be characterized in closed form even for out-of-steady-state dynamics. From (25) and (30) such an equilibrium exists if

$$\omega'(2\bar{h}) \leq 1 + \frac{r}{\alpha} < \omega'\left(\bar{h}\right) \leq \left(1 + \frac{1+r}{\alpha}\right)\left(1 + \frac{r}{\alpha}\right). \tag{40}$$

The economy starts at a steady state at the beginning of $t = 0$. Before entering the DM there is a measure α of buyers holding $m_\ell = \bar{h}M/\left[\alpha\bar{h} + (1-\alpha)z^*\right]$ units of money and a measure $1 - \alpha$ holding $m_h = z^*M/\left[\alpha\bar{h} + (1-\alpha)z^*\right]$. At the beginning of the CM of $t = 0$, after a round of DM trades, the distribution of money balances across buyers has three mass points: there is a measure α of buyers holding no money (the buyers who were matched in the previous DM), a measure $\alpha(1 - \alpha)$ holding m_ℓ and a measure $(1 - \alpha)^2$ holding m_h. This distribution is illustrated in Figure 7.

[7] Alternatively, transfers could be made to both buyers and sellers in which case each buyer receives $(\gamma - 1)M/2$. See Appendix 5.

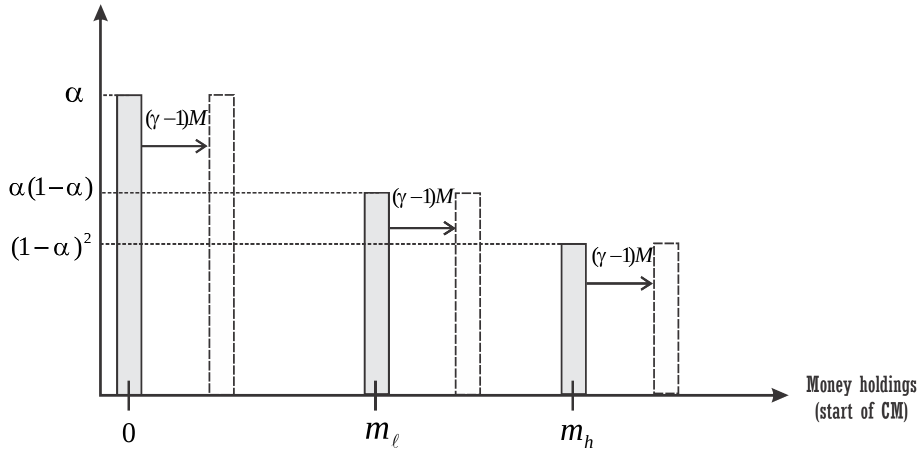

Figure 7: Distribution of money holdings at the start of the CM of $t = 0$

We will construct equilibria such that the economy returns to a steady state in the CM of $t = 1$, i.e.,

$$\phi_t = \phi_1 = \frac{\alpha \bar{h} + (1-\alpha) z^*}{\gamma M} \quad \text{for all } t \geq 1. \tag{41}$$

The short run corresponds to the CM of $t = 0$ and the DM of $t = 1$. We will distinguish small money injections that do not affect the number of mass points in the distributions of real balances from larger money injections such that the distribution F_t is degenerate at $t = 1$.

5.1 Small injection

We consider first the case where γ is close to 1 so that the feasibility constraint on labor, $h \leq \bar{h}$, binds only for the α buyers with no money balances at the beginning of the CM of $t = 0$. We will characterize the distribution of real balances at $t = 1$ and the rate of return of money. We will provide conditions for the proposed equilibrium to exist, and we will study the effects of the money injection on CM and DM output and individual labor supplies.

Distribution of real balances. Consider first buyers who enter the CM of $t = 0$ with no money. Their real balances after transfers and CM trades expressed in the CM good of $t = 1$ are

$$z_1^0 = R_1 \bar{h} + (\gamma - 1)\phi_1 M. \tag{42}$$

The buyer supplies $\bar{h}$ units of labor in the CM of $t = 0$, which yields $R_1 \bar{h}$ real balances at $t = 1$, and he receives a lump-sum transfer of money of size $(\gamma - 1)M$ valued at the price ϕ_1.

We conjecture that the buyers holding m_ℓ and m_h at the beginning of the CM of $t = 0$ accumulate their targeted real balances for period 1, z_1^*, which requires $\xi_0(\phi_0 m_\ell) = \xi_0(\phi_0 m_h) = 0$. We also conjecture that $h \leq \bar{h}$ does not bind for buyers holding z_1^* in the CM of $t = 1$, i.e., $\xi_1(z_1^*) = 0$. From (18) z_1^* solves

$$\omega'(z_1^*) = 1 + \frac{1 + r - R_1}{\alpha R_1}. \tag{43}$$

To summarize, the distribution of real balances at the beginning of $t = 1$ has two mass points: α buyers hold z_1^0 and $1-\alpha$ buyers hold z_1^*. Hence aggregate real balances at $t = 1$ are $\phi_1 \gamma M = \alpha z_1^0 + (1-\alpha) z_1^*$. Substituting z_1^0 by its expression given by (42) and solving for aggregate real balances we obtain:

$$\phi_1 \gamma M = \frac{\alpha R_1 \bar{h} + (1-\alpha) z_1^*}{1 - \alpha \left(1 - \frac{1}{\gamma}\right)}. \tag{44}$$

Aggregate real balances at $t = 1$ are a linear combination of the capitalized labor endowment, $R_1 \bar{h}$, of buyers with depleted money balances and the targeted real balances, z_1^*, of all other buyers. The denominator in (44) takes into account that the α buyers with depleted money balances at $t = 0$ hold onto their transfer of real balances which is equal to a fraction $1 - \gamma^{-1}$ of aggregate real balances.

Rate of return and value of fiat money. Next, we determine the gross real rate of return of fiat money from $t = 0$ to $t = 1$. Our guess that the economy reaches a steady state in the CM of $t = 1$ pins down aggregate real balances, $\phi_1 \gamma M = \alpha \bar{h} + (1-\alpha) z^*$, and hence the rate of return of money. Indeed, from (44),

$$\frac{\alpha R_1 \bar{h} + (1-\alpha) z_1^*}{1 - \alpha \left(1 - \frac{1}{\gamma}\right)} = \alpha \bar{h} + (1-\alpha) z^*. \tag{45}$$

The left side of (45) is increasing in R_1: it is equal to 0 when $R_1 = 0$ and it is greater than $\alpha \bar{h} + (1-\alpha) z^*$ (because $1 > \alpha \left(1 - \gamma^{-1}\right)$) when $R_1 = 1$. Hence, there is a unique R_1 solution to (45) and it is such that $R_1 < 1$ and $\phi_0 > \phi_1$.

Figure 8 illustrates the determination of the equilibrium value for R_1, denoted R_1^e, where the left side of (45) is represented by the upward-sloping red curve. As γ increases this curve shifts upward and, as a result, R_1^e decreases. Moreover, $\lim_{\gamma \downarrow 1} R_1(\gamma) = 1$. So despite prices being flexible and all agents having access to the centralized market where money is injected, the value of money does not adjust instantly to its new steady-state value and money is not neutral in the short run.[8]

Given R_1 we determine the value of money at the time of the money injection, $\phi_0 = \phi_1 / R_1$. Using the expression for ϕ_1 given by (41) we obtain:

$$\phi_0 = \frac{\alpha \bar{h} + (1-\alpha) z^*}{R_1 \gamma M}. \tag{46}$$

The value of money at $t = 0$ decreases with $R_1 \gamma$.

Let $\phi_{-1} = \left[\alpha \bar{h} + (1-\alpha) z^*\right] / M$ denote the value of money at the initial steady state. The ex-post rate of return of money at $t = 0$ is $R_0 = \phi_0 / \phi_{-1} = 1/(R_1 \gamma)$. Next, we determine the condition under which a small increase in γ above 1 raises R_0 above one. In that case the money injection raises the value of money at $t = 0$, ϕ_0, above its initial steady-state value, $\gamma \phi_1$, i.e., there is a deflation in the short run. Differentiating R_1 defined in (45) with respect to γ we show that

$$\left. \frac{dR_1 / R_1}{d\gamma / \gamma} \right|_{\gamma = 1} < -1 \Leftrightarrow \frac{-z^* \omega''(z^*)}{\omega'(z^*)} > \frac{z^*}{\left(z^* - \bar{h}\right) \beta \alpha (\alpha + r)}. \tag{47}$$

[8] Results are qualitatively similar if the money supply increases though transfers to all agents in the economy.

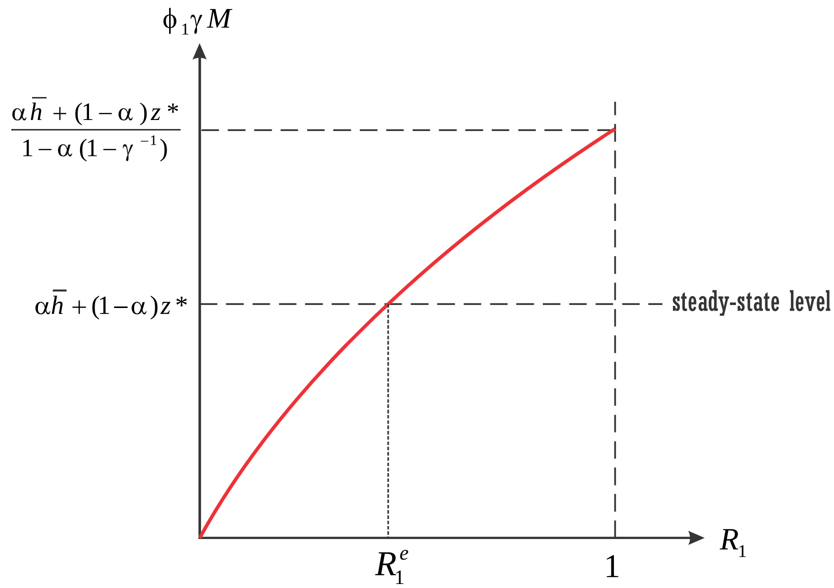

Figure 8: Determination of R_1

If the elasticity of buyers' targeted real balances with respect to R_1 is sufficiently low, which happens when buyers are sufficiently risk averse, then ϕ_0 increases with the size of the money injection. In order to illustrate this possibility we consider a numerical example with the following functional forms and parameter values: $v(y) = y$, $\beta = 0.25$, $\alpha = 0.82$, and $\bar{h} = 0.09$, and

$$u(y) = \frac{A}{1-\eta}\left[\left(\frac{y+b}{A}\right)^{1-\frac{1}{\eta}} - \left(\frac{b}{A}\right)^{1-\frac{1}{\eta}}\right],$$

where $A = 0.1$, $\eta = 0.03$, $b = 10^{-10}$.[9] In Figure 9 we plot ϕ_0 and $\gamma\phi_1$ as a function of the size of the money injection, γ. For low values of γ ($\gamma < 1.5$) the increase in the money supply at time $t = 0$ generates an increase in R_0 (a deflation). In contrast, for large values of γ, R_0 decreases so that the value of money adjusts partially to the increase in the money supply. Finally, there is a value for γ ($\gamma \approx 1.5$) such that $\phi_0 = \gamma\phi_1$ and $R_0 = 1$. The price level in $t = 0$ is the same than the one at the initial steady state. So prices are "sticky" in that they do not adjust at all to a change in the money supply.

Output and labor-supply effects. The output levels in the DM of $t = 1$ are $y_{\ell,1} = v^{-1}(z_1^0) > v^{-1}(\bar{h})$ and $y_{h,1} = v^{-1}(z_1^*) < v^{-1}(z^*)$. Hence, the money injection reduces the dispersion of output and consumption levels across matches. Aggregate output is

$$Y_1 = \alpha y_{\ell,1} + (1-\alpha)y_{h,1} \geq Y^{ss} = \alpha v^{-1}(\bar{h}) + (1-\alpha)v^{-1}(z^*),$$

with a strict inequality if v is strictly convex. So DM aggregate output increases relative to its steady-state value, Y^{ss}. We summarize these results in Figure 10 by plotting the trajectories for aggregate real balances,

[9] The parameter values have ben chosen to maximize the level of deflation over a certain range subject to the constraints for an equilibrium with full depletion and a two-mass-point distribution.

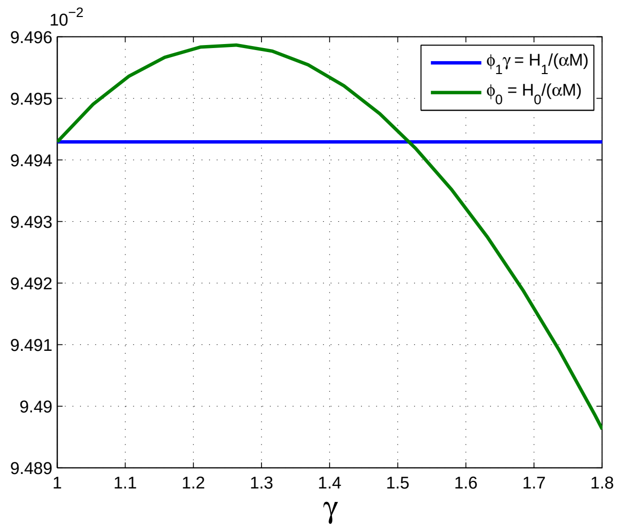

Figure 9: Example where the price level falls below its initial steady-state value in the short-run.

$\phi_t \gamma M$, and DM output levels.

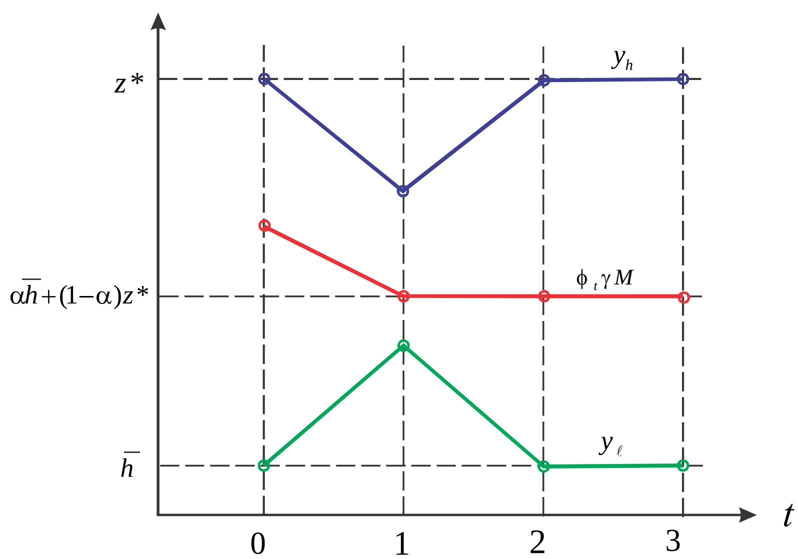

Figure 10: Effects of a small money injection

Next, we turn to CM aggregate output, denoted $H_t \equiv \int \max\{h_t(i), 0\}di$, where $h_t(i)$ is the choice of h at time t by buyer i. Recall that $h_t < 0$ corresponds to consumption of the CM good by the buyer, hence we only count $h_t(i)$ when it is positive. Summing the buyers' budget constraints in the CM of $t = 0$, (5), we find:

$$H_0 = \alpha\bar{h} + \alpha(1-\alpha)h_0^+(m_\ell) + (1-\alpha)^2 h_0^+(m_h), \tag{48}$$

with

$$h_0^+(m_\ell) = \left\{\frac{z_1^*}{R_1} - \phi_0\left[m_\ell + (\gamma-1)M\right]\right\}^+ \tag{49}$$

$$h_0^+(m_h) = \left\{\frac{z_1^*}{R_1} - \phi_0\left[m_h + (\gamma-1)M\right]\right\}^+. \tag{50}$$

The first term on the right side of (48) corresponds to the labor supply of the buyers with depleted money holdings: those buyers supply their labor endowment. The second term corresponds to the labor supply of buyers with m_ℓ units of money, $h_0^+(m_\ell)$, and the third term is the labor supply of buyers with m_h units of money, $h_0^+(m_h)$. From (49) and (50) buyers holding m_ℓ and m_h accumulate their targeted real balances, z_1^*/R_1, and their wealth is composed of their initial real balances and the real transfer of money. By definition of the money holdings at the initial steady state, $\phi_0 m_\ell = \bar{h}/(R_1\gamma)$ and $\phi_0 m_h = z^*/(R_1\gamma)$. Moreover, aggregate real balances are $\phi_0 M = \left[\alpha\bar{h} + (1-\alpha)z^*\right]/(\gamma R_1)$. Substituting these expressions into (49)-(50) and using (45) to express z_1^* as a function of γ and R_1, i.e.,

$$\gamma z_1^* = \frac{\left[(1-\alpha)\gamma + \alpha\right]\left[\alpha\bar{h} + (1-\alpha)z^*\right] - \alpha\gamma R_1\bar{h}}{1-\alpha}, \tag{51}$$

we obtain the following individual labor supplies:

$$h_0^+(m_\ell) = \frac{\left\{\alpha\bar{h}\left(1-\gamma R_1\right) + (1-\alpha)(z^* - \bar{h})\right\}^+}{(1-\alpha)\gamma R_1} \tag{52}$$

$$h_0^+(m_h) = \frac{\alpha\bar{h}\left\{1-\gamma R_1\right\}^+}{(1-\alpha)\gamma R_1}. \tag{53}$$

From (52) and (53) individual labor supplies are decreasing in $\gamma R_1 = \gamma\phi_1/\phi_0$. Moreover, when $\gamma R_1 = 1$ labor supplies are at their steady-state levels. Hence, when $\gamma R_1 < 1$—the value of money at $t = 0$ is larger than the one at the initial steady state—aggregate CM output is larger than its steady-state value. Conversely, when $\gamma R_1 > 1$ it is lower than its steady-state value. So high CM output is associated with deflation while low CM output is associated with inflation. Notice also that when $\gamma R_1 < 1$ all buyers supply some labor in the CM. Since sellers unload all their units of money in the CM, it follows that $H_0 = \alpha\phi_0 M$. In contrast, from (53), when $\gamma R_1 \geq 1$ buyers holding m_h do not supply any labor.

From (47) CM output increases when targeted real balances are inelastic with respect to the rate of return of currency. In that case a money injection induces buyers holding m_ℓ and m_h units of money to supply more labor in the short run (since those buyers are unconstrained by their labor endowment) in order to maintain their targeted real balances, thereby generating a fall in the price level. Indeed, those buyers bid for the αM units of money held by sellers, but this residual money supply might be not be sufficient to allow buyers to reach their target. In that case the value of money in $t = 0$, ϕ_0, rises above its initial steady-state value, $\gamma\phi_1$, in order to clear the money market.

Conditions for the proposed equilibrium. First, we check that $z_1^0 < z_1^*$. From (42)

$$R_1\bar{h} + \left(1 - \frac{1}{\gamma}\right)\left[\alpha\bar{h} + (1-\alpha)z^*\right] < z_1^*. \tag{54}$$

As γ approaches 1 the left side tends to $\bar{h}$ while the right side tends to z^*. Hence, by continuity (54) holds for low values of γ. Substituting z_1^* by its expression given by (51) this condition can be rewritten as:

$$\gamma R_1 < \frac{\alpha \bar{h} + (1-\alpha) z^*}{\bar{h}}. \tag{55}$$

According to (55) the ex-post rate of return of money at time $t = 0$, $R_0 = 1/(R_1 \gamma)$, must be above a threshold less than one. Second, we check that buyers holding m_ℓ units of money at the beginning of the CM of $t = 0$ (before transfers) can accumulate z_1^*, i.e., $h_0^+(m_\ell) \leq \bar{h}$. From (52) this condition can be rewritten as:

$$(1-\alpha)(z^* - \bar{h}) \leq (\gamma R_1 - \alpha)\,\bar{h}. \tag{56}$$

The right side tends to $(1-\alpha)\bar{h}$ as γ goes to 1. Hence, under the condition $z^* \leq 2\bar{h}$, (56) holds for values of γ close to 1. Finally, we check that the $1 - \alpha$ measure of agents with money balances at the beginning of the CM of $t = 1$ can accumulate z^* since $z_1^0 > \bar{h}$. Indeed, from the clearing condition of the money market, $\alpha z_1^0 + (1-\alpha) z_1^* = \alpha \bar{h} + (1-\alpha) z^*$, i.e.,

$$z_1^0 - \bar{h} = \left(\frac{1-\alpha}{\alpha}\right)(z^* - z_1^*) > 0.$$

An equilibrium following a small money injection is described by short-run allocations and prices, (ϕ_0, R_1, z_1^*), that solve (43), (45), (46), (54), and (56), followed by a steady state as characterized above.

We summarize the results of this section in the following proposition.

Proposition 5 ***(Small money injection.)*** *Suppose the economy is initially at a steady state with $N = 2$. A one-time money injection, $(\gamma - 1)M$, in the CM of $t = 0$ that satisfies (54)-(56) has the following consequences:*

1. *It raises aggregate real balances, $\phi_0 \gamma M$, above their steady-state value, and reduces the gross rate of return of money, R_1, below one.*

2. *If (47) holds and γ is close to 1, then $\phi_0 > \gamma \phi_1$, i.e., there is deflation in the short run, and CM output increases, $H_0 > H_1$.*

3. *It generates a mean-preserving reduction in the distribution of real balances in the DM of $t = 1$, an increase in aggregate DM output if $v'' > 0$, and an increase in society's welfare.*

4. *The economy returns to its steady state in the CM of $t = 1$.*

So far we have considered unanticipated increases in the money supply. The empirical evidence regarding monetary shocks is often stated in terms shocks that are contractionary. In the context of our model this would mean $\gamma < 1$, i.e., the monetary authority withdraws $(1-\gamma)M$ through lump-sum taxation. Here we need to assume that the government can enforce the payment of taxes and $1 - \gamma$ is not too large so that households with depleted money balances can afford the tax. The effects of a contraction of the money supply are symmetric to the ones described in Proposition 5.

Corollary 1 ***("Price puzzle")*** *Consider a one-time contraction of the money supply, $\gamma < 1$. If (47) holds, then $\phi_0 < \gamma\phi_1$, i.e., there is inflation in the short run, and CM output decreases, $H_0 < H_1$.*

This finding is consistent with the "price puzzle" from Eichenbaum (1992) according to which a contractionary shock to monetary policy raises the price level in the short run. Moreover, Christiano, Eichenbaum, and Evans (1999, Section 4.4.3) document that a contractionary shock to $M1$ leads output to fall during two quarters and then to rise. They argue that such evidence is hard to reconcile with a sticky price model but it is consistent with our model where the distributional effects of monetary policy create non-neutralities.

More on transfers Proposition 5 has been obtained under the assumption that money transfers were received by buyers only, i.e., the monetary authority can target agents with liquidity needs but it does not make transfers contingent on money holdings. It is easy to show that the parts 1, 3, and 4 of the proposition go through if both buyers and sellers receive the lump-sum transfer. In particular, aggregate real balances, $\phi_0\gamma M$, rise above their steady-state value, which implies that prices are sluggish. However, whether Part 2 of Proposition 5 holds or not, i.e., there can be deflation in the short run, $\phi_0 > \gamma\phi_1$, depends on the size of n. To see this, suppose now that both buyers and sellers receive a transfer. Since buyers represent a fraction $1/(1+n)$ of the population the size of the transfer is $(\gamma-1)M/(1+n)$. The real balances at the beginning of $t = 1$ of the buyers holding $m = 0$ at the time of the money injection are

$$z_1^0 = R_1\bar{h} + (\gamma - 1)\phi_1 \frac{M}{1+n}. \tag{57}$$

The only difference with respect to (42) is the second term on the right side where M is replaced with $M/(1+n)$. Buyers holding m_ℓ and m_h accumulate z_1^* solution to (43). Aggregate real balances at the beginning of $t = 1$ are $\phi_1\gamma M = \alpha z_1^0 + (1-\alpha)z_1^*$. Assuming that the economy returns to its steady state in the CM of $t = 1$ we have:

$$\phi_1\gamma M = \frac{\alpha R_1\bar{h} + (1-\alpha)z_1^*}{1 - \alpha\left(1-\gamma^{-1}\right)/(1+n)} = \alpha\bar{h} + (1-\alpha)z^*. \tag{58}$$

The last term on the right corresponds to aggregate real balances at the steady state. By the same reasoning as before we obtain the following corollary.

Corollary 2 ***(Untargeted transfers.)*** *A one-time money injection, $(\gamma-1)M$, through lump-sum transfers to both buyers and sellers reduces the rate of return of money, $R_1 < 1$, and it generates short-run deflation, $\phi_0 > \gamma\phi_1$, if and only if:*

$$\frac{-z^*\omega''(z^*)}{\omega'(z^*)} > \frac{(1+n)\,z^*}{\left[z^* - \left(1 + \frac{n}{1-\alpha}\right)\bar{h}\right]\beta\alpha\,(r+\alpha)}. \tag{59}$$

Note that the condition for short-run deflation, (59), coincides with the one obtained under the assumption that the money injection goes to buyers only, (47), when $n = 0$. Given that the right side of (59) is increasing in n it follows that a short-run deflation is less likely to occur when the transfer goes to both buyers and sellers. Under the assumption $z^* < 2\bar{h}$ a necessary condition for (59) to hold is $n < 1 - \alpha$. In particular, if

$n = 1$, there is an equal measure of buyers and sellers, then the inequality never holds. So a one-time money injection can generate a deflation in the short run provided that z^* is sufficiently inelastic with respect to R and the transfer is no diluted among a too large measure of sellers.

Next, we disantangle the effects of a money injections according to the recipients of the transfers. Suppose that the transfer is only received by sellers or buyers holding m_ℓ or m_h. In this case the rate of return of money in the short run is determined by:

$$\alpha R_1 \bar{h} + (1-\alpha) z_1^* = \alpha \bar{h} + (1-\alpha) z^*.$$

Relative to (44) the denominator of the left side is equal to one because the poorest buyers with depleted money balances do not receive any money. It is easy to check that $R_1 = 1$, i.e., money is neutral. As long as money is not transferred to the buyers for whom $h \leq \bar{h}$ binds, then the money injection has no real effect. Hence, in the experiments described above, the real effects of a change in the money supply only arise through the transfer to the buyers with depleted money holdings.

Let us consider next a money injections through transfers to buyers with depleted money balances. Such transfers require that the monetary authority can observe the money holdings of buyers. If buyers can hide their money, such a transfer would not be incentive compatible. Each one of the α buyers with depleted money balances at the beginning of the CM receives $(\gamma - 1)M/\alpha$. The rate of return of money in the short-run solves:

$$\alpha \gamma R_1 \bar{h} + (1-\alpha) \gamma z_1^* = \alpha \bar{h} + (1-\alpha) z^*.$$

It is easy to check that $R_1 < 1$. So money is no longer neutral. The following corollary summarizes this discussion.

Corollary 3 ***(Targeted transfers.)*** *A one-time money injection, $(\gamma - 1)M$, through transfers to sellers or buyers holding m_ℓ or m_h is neutral, i.e., $R_1 = 1$. In contrast, if money is only received by the buyers with depleted money holdings, then $R_1 < 1$ and short-run deflation, $\phi_0 > \gamma \phi_1$, occurs if and only if:*

$$\frac{-\omega''(z^*) z^*}{\omega'(z^*)} > \frac{1}{\beta(r+\alpha)}. \tag{60}$$

5.2 Large injections

We now consider the case of a large money injection. Suppose that all buyers enter the DM of period 1 with z_1^* real balances where z_1^* solves (43), i.e., the distribution of real balances is degenerate. Assuming that the economy has reached its steady state in the CM of $t = 1$ it follows that

$$z_1^* = \alpha \bar{h} + (1-\alpha) z^*. \tag{61}$$

The right side of (61), z_1^*, tends to 0 as R_1 approaches 0 and it is equal to z^* when $R_1 = 1$. Hence, (61) determines a unique $R_1 < 1$, which is independent of γ. So an increase in the size of the money injection affects current and future prices in the same proportion so as to keep their ratio, ϕ_1/ϕ_0, constant. As before, the money injection generates a mean-preserving decrease in the spread of the distribution of real balances across buyers. Aggregate output in the DM of $t = 1$ is

$$Y_1 = v^{-1}\left[\alpha\bar{h} + (1-\alpha)z^*\right] \geq Y^{ss} = \alpha v^{-1}(\bar{h}) + (1-\alpha)v^{-1}(z^*).$$

It is independent of the size of the money injection, but it is larger than the steady-state value provided that $v'' > 0$.

We need to check that the buyers who enter the CM of $t = 0$ with no money balances are not constrained by their endowment of labor. This will be the case if $R_1\bar{h} + \left(1 - \frac{1}{\gamma}\right)\phi_1\gamma M > z_1^*$, i.e.,

$$\gamma R_1 > \frac{\alpha\bar{h} + (1-\alpha)z^*}{\bar{h}}. \tag{62}$$

So an equilibrium with a degenerate distribution of real balances at $t = 1$ exists provided that the size of the transfer is sufficiently large. Moreover, from (62) the rate of return of money is $R_1 > \gamma^{-1}$. So for large money injections it is always the case that $\phi_0 < \left[\alpha\bar{h} + (1-\alpha)z^*\right]/M$, prices increase relative to their initial steady-state value. As a result, from (53), buyers holding m_h do not supply any labor. Buyers holding no money supply

$$\begin{aligned} h_0^+(0) &= \frac{z_1^*}{R_1} - (\gamma - 1)\phi_0 M \\ &= \frac{\alpha\bar{h}\left(1 - \gamma R_1\right) + (1-\alpha)z^*}{(1-\alpha)\gamma R_1}. \end{aligned} \tag{63}$$

From (52) and (63) both $h_0^+(0)$ and $h_0^+(m_\ell)$ decrease with γR_1. Using that $\gamma R_1 > 1$ it follows that aggregate output in the CM of $t = 0$ is lower than its steady-state value.

Proposition 6 ***(Large money injection.)*** *Suppose the economy is initially at a steady state with $N = 2$. A large one-time money injection, $(\gamma - 1)M$, in the CM of $t = 0$ such that (62) holds has the following consequences:*

1. *It raises aggregate real balances, $\phi_0\gamma M$, above their steady-state value, and reduces the gross rate of return of money, R_1, below one. Moreover, $\phi_0 < \gamma\phi_1$ and $H_0 < H_1$.*
2. *The distribution of real balances is degenerate in the DM of $t = 1$ and DM aggregate output is higher than its steady-state value if $v'' > 0$.*
3. *The economy returns to its steady state in the CM of $t = 1$.*

6 Long-lasting effects of a one-time money injection

So far we have restricted our attention to equilibria with $N = 2$ because such equilibria generate simple, one-period transitions to a steady state. We now consider steady states with $N \geq 3$ and we assume a small injection of money so that the distribution of real balances preserves N mass points with probabilities given by (29)-(30). We will focus on equilibria with full depletion.

Consider period $t \geq N$. Buyers who have not been matched in any DM since the money injection in $t = 0$ had the possibility to reach their targeted real balances. Hence, aggregate real balances in period t are defined by

$$\phi_t \gamma M = \sum_{j=1}^{N-1} \mu_j \bar{h} \left(\sum_{n=1}^{j} \frac{\phi_t}{\phi_{t-n}} \right) + \mu_N z_t^*, \text{ for all } t \geq N. \tag{64}$$

In order to explain the right side of (64) recall that we can index buyers by the last time they had a DM encounter and depleted their money balances. There is a measure μ_1 of buyers who entered the CM of $t-1$ with depleted money balances. Those buyers accumulated $\bar{h}$ real balances and entered the period t with $R_t \bar{h} = (\phi_t/\phi_{t-1})\bar{h}$ real balances. There is a measure μ_2 of buyers who entered the CM of $t-2$ with no money balances and who where unmatched in the DM of $t-1$. Such buyers accumulated $\bar{h}$ in the CM of $t-2$ and the CM of $t-1$ so that their real balances at the beginning of t are $R_{t-1}R_t\bar{h} + R_t\bar{h} = (\phi_t/\phi_{t-2} + \phi_t/\phi_{t-1})\bar{h}$. We have the same reasoning for buyers who entered the CM of $t-n$ with no money and did not trade in all subsequent DMs until the beginning of t, where $n \leq N-1$. Finally, there is a measure μ_N of buyers who entered the CM of $t-N$ with no money and did not get matched in the DMs of $t-N+1$ until the DM of $t-1$. Those buyers whose labor supply is unconstrained accumulate z_t^* solution to

$$\omega'(z_t^*) = 1 + \frac{1 - \beta R_t}{\alpha \beta R_t}. \tag{65}$$

Let us turn to periods $t < N$. Buyers hold onto the transfer of money they received in the CM of $t = 0$ until they reach their target. Hence, aggregate real balances are

$$\begin{aligned} \phi_t \gamma M \quad = \quad & \sum_{j=1}^{t-1} \mu_j \bar{h} \left(\sum_{n=1}^{j} \frac{\phi_t}{\phi_{t-n}} \right) + \mu_N z_t^* \\ & + \sum_{j=t}^{N-1} \mu_j \left\{ \phi_t \left[m_{j-t} + (\gamma - 1)M \right] + \bar{h} \sum_{n=1}^{t} \frac{\phi_t}{\phi_{t-n}} \right\}, \end{aligned} \tag{66}$$

for all $t \leq N-1$ where

$$m_k = \frac{\alpha k \bar{h} M}{\bar{h}\left\{1 - (1-\alpha)^{N-1}\left[(N-1)\alpha + 1\right]\right\} + \alpha(1-\alpha)^{N-1} z^*}.$$

The first term on the right side of (66) corresponds to buyers who have been matched since the money transfer took place. There is a measure μ_j of buyers who had their last match in the DM of $t - j \geq 1$. Such buyers accumulated $\bar{h}$ real balances in every CM from $t-j$ to $t-1$. The value of the real balances accumulated in the CM of $t-n$ are $\bar{h}(R_{t-n+1} \times ... \times R_t) = \bar{h}(\phi_t/\phi_{t-n})$. The second term on the right side of (66) corresponds to buyers who have reached their target. The novelty relative to (64) is the third term on the right side of (66). For all j between t and $N-1$ there is a measure μ_j of buyers who entered the CM of $t = 0$ with $m_{j-t} = (j-t)\bar{h}/\phi_{-1}$ units of money, where ϕ_{-1} is the initial steady-state value given by (31). Such buyers have not reached their targeted real balances. So they kept their initial wealth, composed of the initial m_j units of money and the transfer $(\gamma - 1)M$, and they accumulated $\bar{h}$ real balances in the following t consecutive periods.

A candidate equilibrium is a sequence $\{\phi_t\}_{t=0}^{+\infty}$ that satisfies (64)-(66). In order to guarantee that we have an equilibrium we check that $h \leq \bar{h}$ binds for the buyer of type $N-1$ but not for the type N. These conditions can be written as

$$\phi_t m_{N-1-t} < z_t^* - \bar{h}\sum_{n=1}^{t}\frac{\phi_t}{\phi_{t-n}} - (\gamma-1)\phi_t M \leq \phi_t m_{N-t}, \ \ \forall t \leq N-1 \tag{67}$$

$$\bar{h}\sum_{n=1}^{N-1}\frac{\phi_t}{\phi_{t-n}} < z_t^* \leq \bar{h}\sum_{n=1}^{N}\frac{\phi_t}{\phi_{t-n}}, \ \ \forall t \geq N. \tag{68}$$

We must also check that buyers have incentives to deplete their money holdings in full at all point in time, i.e.,

$$1 + \frac{1-\beta R_t}{\alpha\beta R_t} \geq \lambda_t(0) \ \text{ for all } t, \tag{69}$$

where, from (18),

$$\begin{aligned} \lambda_t(0) \ &= \ \sum_{n=1}^{N-1}\alpha(1-\alpha)^{n-1}\beta^n\frac{\phi_{t+n}}{\phi_t}\omega'\left[\bar{h}\sum_{u=1}^{n}\frac{\phi_{t+n}}{\phi_{t+n-u}}\right] \\ &\quad +(1-\alpha)^{N-1}\beta^{N-1}\frac{\phi_{t+N-1}}{\phi_t}. \end{aligned} \tag{70}$$

It can be seen from (64) and (66) that the transition to a steady state is long lived because the real balances of a buyer whose last match was in $t-j$, with $j < N-1$, depends on the sequence of past rates of return, R_{t-j+1} to R_t.

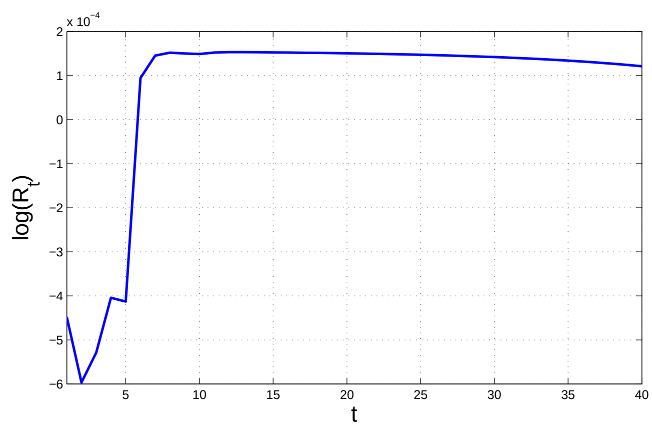

Figure 11: Transitional dynamics following a one-time money injection

In Figure 11 we provide a numerical example for the transitional dynamics of the rate of return of money, R_t, following a one-time money injection. We adopt the following parameter values: $\bar{h} = 0.095$, $\alpha = 0.1$, $u(c) = 2\sqrt{c}$, $r = 0.04$, $\gamma = 1.05$. For such parametrization it takes $N = 6$ periods for buyers with depleted money balances to reach the targeted real balances. For the first six periods following the increase in the money supply the net rate of return of money, $R_t - 1$, is negative, i.e., the value of money declines over time. The fall in the value of money overshoots its new steady-state value. As a result for all $t \geq 7$ the rate of return of money is positive (but small). So the initial periods of inflation are followed by long-lasting periods of deflation. The economy returns to its steady state asymptotically.

7 Money and unemployment risk

So far we have considered an environment where the only idiosyncratic risk comes from random matching in the decentralized goods market. We now extend the model to add idiosyncratic employment shocks formalized as shocks on labor endowments, $\bar{h} \in \{\bar{h}^U, \bar{h}^E\}$, with $0 < \bar{h}^U < \bar{h}^E$.[10] One can think of $\bar{h}^E$ as the income of an employed worker and $\bar{h}^U$ the income of an unemployed worker. The transition from E (employment state) to U (unemployment state) occurs at the beginning of a period with probability s (think of the separation rate) while the transition from U to E occurs with probability f (think of the job finding rate).[11] The presence of these employment/unemployment states will allow us to analyze the effects of money injections as a function of the equilibrium unemployment rate.

We focus on equilibria with full depletion of real balances. We define $\mu^{\chi}_{n,m}$ as the measure of buyers who went trough n periods of employment (state E in the CM) and m periods of unemployment (state U in the CM) since their last DM match and whose current employment state upon entering the DM is $\chi \in \{E, U\}$. These buyers hold $n\bar{h}^E + m\bar{h}^U$ real balances. Let $\mathbb{H} = \{(n, m) \in \mathbb{N}_0^2 : n\bar{h}_E + m\bar{h}_U < z^*\}$ be the set of employment histories, (n, m), for which buyers have not reached their target. The distribution of buyers across states is defined recursively as follows:

$$\mu^E_{n,m} = (1-s)\mu^E_{n-1,m} + f\mu^U_{n,m-1} \tag{71}$$

$$\mu^U_{n,m} = s\mu^E_{n-1,m} + (1-f)\mu^U_{n,m-1}, \tag{72}$$

for all $(n, m) \in \mathbb{H}$ and $n, m > 0$. According to (71) a buyer is in state (n, m, E) if: (i) he was in state $(n-1, m, E)$ in the past period, in which case he enjoyed an endowment of $\bar{h}_E$ in his last CM, and he remained employed with probability $1-s$; (ii) he was in state $(n, m-1, U)$ in the past period, in which case he enjoyed an endowment of $\bar{h}_U$ in his last CM, and he became employed with probability f. Equation (72) has a similar interpretation. If $n = 0$ or $m = 0$ we have:

$$\mu^E_{1,0} = (1-s)\mu^E_{0,0}, \quad \mu^E_{0,1} = f\mu^U_{0,0} \tag{73}$$

$$\mu^U_{0,1} = (1-f)\mu^U_{0,0}, \quad \mu^U_{1,0} = s\mu^E_{0,0}, \tag{74}$$

where $\mu^{\chi}_{0,0}$ is measure of agents with depleted money holdings who are in labor state $\chi \in \{E, U\}$. Because all buyers, irrespective of their state, deplete their money holdings in full in a DM match, $\mu^E_{0,0}$ is equal to the fraction of buyers in state E in the whole population, denoted μ^E, and $\mu^U_{0,0}$ is the fraction of buyers in

[10] The description of the employment shocks is analogous to the one in Algan, Challe, and Ragot (2011). We differ from this model in that we consider an environment with search and bargaining where the idiosyncratic risk comes from both matching opportunities in a decentralized market and employment shocks in a centralized market.

[11] The description of employment/unemployment is similar to the one in Ljungqvist and Sargent (1998) where wages are drawn from an exogenous distribution. One could elaborate on this theory of unemployment by endogenizing wage offers and job finding rates as in standard search models of the labor market.

state U in the whole population, denoted μ^U. At the steady state $\left(1-\mu^U\right)s=\mu^U f$ so that the measure of unemployed buyers is $\mu^U=s/(s+f)$. Hence,

$$\mu_{0,0}^{E}=\frac{f}{s+f},\quad \mu_{0,0}^{U}=\frac{s}{s+f}. \tag{75}$$

Finally, the measures of buyers at the targeted real balances are given by:

$$\mu_{z^*}^{E} = \frac{f}{s+f}-\sum_{(n,m)\in\mathbb{S}}\mu_{n,m}^{E} \tag{76}$$

$$\mu_{z^*}^{U} = \frac{s}{s+f}-\sum_{(n,m)\in\mathbb{S}}\mu_{n,m}^{U}. \tag{77}$$

Equations (71)-(77) define the distribution of buyers across states, $\{\mu_{n,m}^{\chi},\mu_{z^*}^{\chi}\}$, recursively.

We can now compute aggregate real balances:

$$\phi M=\sum_{(n,m)\in\mathbb{S}}\left(\mu_{n,m}^{E}+\mu_{n,m}^{U}\right)\left(n\bar{h}^{E}+m\bar{h}^{U}\right)+\left(\mu_{z^*}^{E}+\mu_{z^*}^{U}\right)z^*. \tag{78}$$

The value of money increases with the income in the two employment states, $\bar{h}^E$ and $\bar{h}^U$, it increases with the job finding rate, f, and it decreases with the separation rate, s. So the value of money is negatively correlated with the unemployment rate.

The marginal value of real balances for a buyer in state E solves

$$\lambda^{E}(z)=\beta\left\{\alpha\omega'\left(z+\bar{h}^{E}\right)+(1-\alpha)\left[(1-s)\lambda^{E}(z+\bar{h}^{E})+s\lambda^{U}(z+\bar{h}^{E})\right]\right\}, \tag{79}$$

for all $z\leq z^*-\bar{h}^E$. For all $z\in[z^*-\bar{h}^E,z^*]$, $\lambda^E(z)=1$. The interpretation of (79) is similar to the one we had before except for the transitions between different employment states. Similarly, the marginal value of real balances for a buyer in state U solves

$$\lambda^{U}(z)=\beta\left\{\alpha\omega'\left(z+\bar{h}^{U}\right)+(1-\alpha)\left[(1-f)\lambda^{U}(z+\bar{h}^{U})+f\lambda^{E}(z+\bar{h}^{U})\right]\right\}, \tag{80}$$

for all $z\leq z^*-\bar{h}^U$. For all $z\in[z^*-\bar{h}^U,z^*]$, $\lambda^U(z)=1$. Using that $\lambda^U(z)\geq\lambda^E(z)$ for all z, the condition for full depletion of real balances is

$$\lambda^{U}(0)\leq\omega'\left(z^*\right)=1+\frac{r}{\alpha}. \tag{81}$$

As before, this condition will hold provided that agents are sufficiently impatient.

In order to keep the analysis very tractable suppose now that employed buyers can reach the targeted real balances in a single period while unemployed workers need two periods to accumulate z^*, i.e.,

$$\bar{h}_U<\omega'^{-1}\left(1+\frac{r}{\alpha}\right)\leq\min\{2\bar{h}_U,\bar{h}_E\}.$$

The marginal value of real balances for an employed buyer is $\lambda_E(z)=1$ for all z on the support of the distribution. So employed buyers are similar to the agents in the Lagos-Wright model: their choice of real balances is unaffected by their labor endowment. From (80) the marginal value of real balances for an unemployed buyer solves

$$\lambda_U(z)=\beta\left\{\alpha\omega'\left(z'\right)+(1-\alpha)\left[f+(1-f)\lambda_U(z')\right]\right\}, \tag{82}$$

where $z' = \min\{z + \bar{h}_U, z^*\}$. The condition for full depletion of real balances, (81), can be reexpressed as

$$\frac{r}{\alpha} > \alpha\beta \left[\omega'\left(\bar{h}_U\right) - \omega'(z^*)\right]. \tag{83}$$

So the endowment of the unemployed, $\bar{h}_U$, cannot be too low—in particular, it cannot be zero—since otherwise the buyer would not want to deplete his money holdings in full when matched in the DM. From (78) the steady-state aggregate real balances simplify to

$$\phi M = \alpha\mu^U \bar{h}^U + \left(1 - \alpha\mu^U\right) z^*. \tag{84}$$

The value of money increases with the income of unemployed workers, $\bar{h}_U$, and it decreases with the unemployment rate, μ^U.[12]

Consider a small money injection through a lump-sum transfer to all buyers. (The policy maker cannot distinguish employed from unemployed buyers.) Unemployed buyers who had depleted their real balances in $t = 0$ enter the following period with

$$z_1^0 = R_1\bar{h}^U + (\gamma - 1)\phi_1 M.$$

The real balances of the unemployed buyer who depleted his money holdings in the last DM corresponds to his income when unemployed times the rate of return of currency, $R_1\bar{h}^U$, plus the real transfer of money, $(\gamma - 1)\phi_1 M$. This transfer is analogous to some unemployment insurance financed with a proportional on money holdings. Following the same reasoning as above the rate of return of money solves

$$\frac{\alpha\mu^U R_1\bar{h}^U + \left(1 - \alpha\mu^U\right) z_1^*}{1 - \alpha\mu^U\left(1 - \frac{1}{\gamma}\right)} = \alpha\mu^U\bar{h}^U + \left(1 - \alpha\mu^U\right) z^*. \tag{85}$$

So the effect of a money injection on the rate of return of money depends on the unemployment rate as represented by μ^U. If μ^U is small, then money is almost neutral whereas if μ^U is large a money injection has real effects.

8 Optimal inflation and unemployment

So far we have described a one-time, unanticipated change in the quantity of money.[13] We now turn to the case where the government implements a constant growth of the money supply by injecting $M_{t+1} - M_t = (\gamma - 1)M_t$ at the beginning of each CM, where γ is close to 1, through lump-sum transfers to buyers only. As in Section 7 the labor endowment of the buyer follows a two-state Markov chain, $\bar{h} \in \{\bar{h}^U, \bar{h}^E\}$. This extension allows us to study the relationship between the optimal inflation rate and the unemployment rate.

[12] The chanel through which unemployment affects the value of money is different from the one in Berentsen, Menzio, and Wright (2011) where $\bar{h}_U$ is large enough to allow unemployed workers to accumulate z^*. In that model it is assumed that α is an increasing function of aggregate employment, $1 - \mu^U$, so that an increase in μ^U reduces α and hence z^*.

[13] More precisely, we have assumed that agents do not assign a positive probability to a change in the money supply ahead of time. However, when the change in the money supply happens it is common-knowledge. In the Appendix A6 we consider the case where a one-time money injection is announced one period ahead. The results are essentially unaffected except for the fact that the early annoucement triggers real effects ahead of the actual money injection.

We focus on steady-state equilibria where aggregate real balances are constant. The gross rate of return of money is $R = \gamma^{-1}$. Generalizing (18) the marginal value of real balances solves

$$\lambda^{\chi}(z) = \frac{\beta}{\gamma}\left\{\alpha\omega'(z') + (1-\alpha)\mathbb{E}\lambda^{\chi'}(z')\right\}, \quad \chi \in \{U, E\}, \tag{86}$$

where the expectation is with respect to the future employment state, $\chi' \in \{U, E\}$, conditional on the current employment state, χ. At the target $z = z' = z^*_\gamma$ with $\lambda^{\chi}(z^*) = 1$ for all $\chi \in \{U, E\}$. Hence, from (86) z^*_γ solves

$$\omega'(z^*_\gamma) = 1 + \frac{\gamma - \beta}{\beta\alpha}. \tag{87}$$

As a benchmark, suppose first that $\bar{h}^E > \bar{h}^U > z^*$, which will be the case if $\omega'(\bar{h}^U) < 1 + (\gamma - \beta)/\beta\alpha$. All buyers, irrespective of their labor state, can accumulate their targeted real balances in a single CM as it is the case in Berentsen, Menzio, and Wright (2010). As γ rises above one, aggregate real balances, $Z \equiv \phi_t M_t = z^*_\gamma$, CM and DM output, and welfare, $\mathcal{W}_\gamma = \alpha\left[\omega(z^*_\gamma) - z^*_\gamma\right]$, decrease. Indeed, there is no trade-off for monetary policy between risk sharing and self insurance since there is no ex-post heterogeneity in terms of real balances. Hence, monetary policy should only promote self insurance by raising the rate of return of currency.[14]

Proposition 7 ***(Constant money growth with degenerate distributions.)*** *Consider a laissez-faire equilibrium with $\gamma = 1$ and $\bar{h}^E > \bar{h}^U > z^*$. A small increase of γ reduces aggregate real balances, output, and social welfare.*

For the rest of the section we study equilibria where buyers in state E have enough endowment to reach their target, $\bar{h}^E > z^*$, while buyers in state U need two rounds of CM to reach z^*, $\bar{h}^U < z^* < 2\bar{h}^U$. At the beginning of each period the distribution of real balances has two mass points. There is a measure $\alpha\mu^U$ of buyers holding $z_\ell \equiv \gamma^{-1}\bar{h}^U + (\gamma - 1)\phi_t M_{t-1}$. Those unemployed buyers depleted their money holdings in the previous DM, they supplied their full labor endowment in the CM, and they held onto the money transfer provided by the government. There is a measure $1 - \alpha\mu^U$ of buyers holding their targeted real balances, z^*_γ. Aggregate real balances are

$$Z \equiv \phi_t M_t = \alpha\mu^U\left[\frac{\bar{h}^U}{\gamma} + (1 - \gamma^{-1})\phi_t M_t\right] + (1 - \alpha\mu^U)z^*_\gamma. \tag{88}$$

Solving for Z we obtain:

$$Z = \frac{\alpha\mu^U\gamma^{-1}\bar{h}^U + (1 - \alpha\mu^U)z^*_\gamma}{\alpha\mu^U\gamma^{-1} + 1 - \alpha\mu^U}. \tag{89}$$

Aggregate real balances (and CM output) are a weighted average of the buyer's labor endowment when unemployed and his targeted real balances, where the weights vary with the money growth rate and the unemployment rate. As γ increases the weight on $\bar{h}^U$ decreases, which tends to raise real balances, but the targeted real balances decrease, which tends to reduce aggregate real balances. As the unemployment rate

[14] In Berentsen, Menzio, and Wright (2010) inflation reduces firm entry and hence can raise welfare in the presence of congestion externality.

increases, aggregate real balances decrease. From (89) the real balances of the poorest unemployed buyers, z_ℓ, are

$$z_\ell = \frac{\gamma^{-1}\bar{h}^U + (1-\gamma^{-1})(1-\alpha\mu^U)z^*_\gamma}{\alpha\mu^U\gamma^{-1} + 1 - \alpha\mu^U}. \tag{90}$$

If the unemployment rate is higher, then aggregate real balances are lower and the lump-sum transfer to unemployed buyers with depleted real balances is lower, which reduces their real balances. Differentiating z_ℓ we obtain

$$\left.\frac{dz_\ell}{d\gamma}\right|_{\gamma=1} = (1-\alpha\mu^U)\left(z^* - \bar{h}^U\right) > 0.$$

For low inflation rates z_ℓ increases with γ while z^*_γ decreases with γ. This effect corresponds to the redistributional role of inflation. The overall effect on aggregate real balances is given by:

$$\left.\frac{\partial Z}{\partial\gamma}\right|_{\gamma=1} = (1-\alpha\mu^U)\left[\alpha\mu^U\left(z^* - \bar{h}^U\right) + \frac{1}{\beta\alpha\omega''(z^*)}\right].$$

A small inflation raises aggregate real balances if the following inequality holds:

$$\frac{-\omega''(z^*)z^*}{\omega'(z^*)} > \frac{z^*}{(r+\alpha)\,\beta\alpha\mu^U\left(z^* - \bar{h}^U\right)}. \tag{91}$$

If ω' is very elastic then a change in γ does not affect buyers' target much but it raises the real balances of the poorest buyers by the amount of the lump-sum transfer. In this case a small increase in γ generates an increase in the mean of the distribution of real balances and a decrease in its dispersion. Hence, welfare increases. Note also that the condition (91) is more likely to hold when the unemployment rate is high since the measure of buyers who are unable to reach the target increases with μ^U.

We define society's welfare at a steady state with money growth rate γ by

$$\mathcal{W}_\gamma = \alpha^2\mu^U\left[\omega(z_\ell) - z_\ell\right] + (1-\alpha\mu^U)\alpha\left[\omega(z^*_\gamma) - z^*_\gamma\right]. \tag{92}$$

The first term on the right side of (92) corresponds to matches between an unemployed buyer holding z_ℓ and a seller. There is a measure $\alpha\mu^U$ of such buyers and each of them has a probability α of being matched, so the total number of matches is $\alpha^2\mu^U$. The second term of the welfare function corresponds to matches between buyers holding their targeted real balances—there is a measure $1-\alpha\mu^U$ of such buyers—and sellers. We differentiate $\mathcal{W}_\gamma$ in the neighborhood of a constant money supply to obtain:

$$\left.\frac{d\mathcal{W}_\gamma}{d\gamma}\right|_{\gamma=1} = (1-\alpha\mu^U)\left\{\alpha^2\mu^U\left[\omega'(\bar{h}^U) - 1\right]\left(z^* - \bar{h}^U\right) + \frac{r}{\beta\alpha\omega''(z^*)}\right\}.$$

Hence, inflation is welfare improving if

$$\frac{-\omega''(z^*)z^*}{\omega'(z^*)} > \frac{r}{(r+\alpha)\beta\alpha^2\mu^U\left[\omega'(\bar{h}^U) - 1\right]}\left(\frac{z^*}{z^* - \bar{h}^U}\right). \tag{93}$$

A positive inflation rate is more likely to be optimal when the unemployment rate, μ^U, is high or when the labor income of the unemployed, $\bar{h}^U$, is low. Indeed, the risk sharing benefits of inflation are larger when the income of the unemployed, $\bar{h}^U$, is far away from their desired level of insurance, z^*, and when the measure of buyers with low labor endowments, μ^U, is large.

Proposition 8 ***(Optimal inflation and unemployment.)*** *Consider a laissez-faire equilibrium with $\gamma = 1$, $z^* < \bar{h}^E$, and $\bar{h}^U < z^* < 2\bar{h}^U$. Moreover, assume that the equilibrium features full depletion of money holdings, i.e., (83) holds. Anticipated inflation through lump-sum transfers to buyers raises aggregate real balances if (91) holds and it raises social welfare if (93) holds.*

9 Conclusion

We constructed a tractable of monetary exchange with search and bargaining featuring a non-degenerate distribution of real balances that can be used to study the short-run and long-run effects of monetary policy. We adopted the New-Monetarist environment of Lagos and Rocheteau (2005) and studied the equilibrium set when the feasibility constraint on labor supply, $h \leq \bar{h}$, binds. While value functions are no longer linear and the equilibrium distribution of real balances is non-degenerate the model remains solvable in closed form for a large set of parameter values and it can easily be solved numerically for other parameter values. The model generates new insights for short-run and long-run effects of money. A one-time injection of money in a centralized market with flexible prices and unrestricted participation leads to higher aggregate real balances in the short run and, under some conditions, higher aggregate output. The effects on the rate of return of money and prices are non-monotone with the size of the money injection. We provided examples where money non-neutralities are long lived and non-monotone over time. Finally, we studied a version of the model with both random-matching risk and employment risk. We showed that a constant money growth rate can lead to higher output and welfare if the unemployment rate is large and agents are sufficiently risk averse.

References

[1] Algan, Yann, Edouard Challe, and Lionel Ragot (2011). "Incomplete markets and the output-inflation trade-off," Economic Theory 46, 55-84.

[2] Bewley, Truman (1980). "The optimum quantity of money," in J. H. Kareken and N. Wallace (eds), Models of Monetary Economies, Federal Reserve Bank of Mineapolis, 169-210.

[3] Bewley, Truman (1983). "A difficulty with the optimum quantity of money," Econometrica 51, 1485-1504.

[4] Camera, Gabriele, and Dean Corbae (1999). "Money and price dispersion," International Economic Review 40, 985-1008.

[5] Chiu, Jonathan, and Miguel Molico (2010). "Liquidity, redistribution, and the welfare cost of inflation," Journal of Monetary Economics 57, 428-438.

[6] Chiu, Jonathan, and Miguel Molico (2011). "Uncertainty, inflation, and welfare," Journal of Money, Credit, and Banking, 43, 487-512.

[7] Chiu, Jonathan, and Miguel Molico (2014). "Short-run dynamics in a search-theoretic model of monetary exchange," Working Paper.

[8] Christiano, Lawrence , Martin Eichenbaum, Charles Evans (1999). "What have we learned and to what end?" in Handbook of Macroeconomics, ed. Michael Woodford and John D. Taylor.

[9] Coibion, Olivier, Yuriy Gorodnichenko, Lorenz Kueng and John Silvia (2012). "Innocent bystanders? Monetary policy and inequality in the U.S.," Working Paper.

[10] Doepke, Matthias, and Martin Schneider (2006). "Inflation and the redistribution of nominal wealth," Journal of Political Economy 114, 1069-1097.

[11] Eichenbaum, Martin (1992) "Comment on interpreting the macroeconomic time series facts: The effects of monetary policy," European Economic Review 36, 1001-1011.

[12] Faig, Miquel, and Zhe Li (2009). "The welfare costs of expected and unexpected inflation." Journal of Monetary Economics 56, 1004-1013.

[13] Friedman, Milton and Anna Schwartz (1963). A Monetary History of the United States: 1867-1960, National Bureau of Economic Research.

[14] Green, Edward and Ruilin Zhou (1998). "A rudimentary random-matching model with divisible money and prices," Journal of Economic Theory 81, 252–271.

[15] Green, Edward and Ruilin Zhou (2002). "Dynamic monetary equilibrium in a random-matching economy," Econometrica, 70, 929-969.

[16] Jin, Gu, and Tao Zhu (2014). "Non neutrality of money in dispersion: Hume revisited," Working Paper.

[17] Lagos, Ricardo and Guillaume Rocheteau (2005). "Inflation, output and welfare," International Economic Review 46, 495-522.

[18] Lagos, Ricardo and Randall Wright (2005). "A unified framework for monetary theory and policy analysis," Journal of Political Economy 113, 463–484.

[19] Lagos, Ricardo, Guillaume Rocheteau, and Randall Wright (2015). "Liquidity: A new-monetarist perspective," Journal of Economic Literature (Forthcoming).

[20] Ljungqvist, Lars, and Thomas Sargent (1998). "The European unemployment dilemma," Journal of Political Economy 106, 514-550.

[21] Menzio, Guido, Shouyong Shi, and Hongfei Sun (2013), "Monetary theory with non-degenerate distributions", Journal of Economic Theory 148, 2266-2312.

[22] Molico, Miguel (2006). "The distribution of money and prices in search equilibrium," International Economic Review 47, 701-722.

[23] Ramey, Valerie (2015). "Macroeconomic shocks and their propagation," mimeo.

[24] Rocheteau, Guillaume, Pierre-Olivier Weill, and Tsz-Nga Wong (2015). "A tractable model of monetary exchange with ex-post heterogeneity," NBER Working Paper 21179.

[25] Rocheteau, Guillaume, and Randall Wright (2005). "Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium," Econometrica 73, 175-202.

[26] Scheinkman, Jose, and Laurence Weiss (1986). "Borrowing constraints and aggregate economic activity", Econometrica 54, 23-45

[27] Sims, Christopher (1972). "Money, income, and causality," American Economic Review 62, 540-552.

[28] Williamson, Stephen (2006). "Search, limited participation, and monetary policy," International Economic Review 47, 107-128.

[29] Zhu, Tao (2005). "Existence of a monetary steady state in a matching model: Divisible money", Journal of Economic Theory 123, 135-160.

[30] Zhu, Tao (2008). "An overlapping-generations model with search," Journal of Economic Theory 142, 318-331.

Appendix A1: Proofs of Lemmas and Propositions

Proof of Lemma 1 From (8) we define a mapping $T : \mathcal{B}(\mathbb{R}_+ \times \mathbb{N}) \to \mathcal{B}(\mathbb{R}_+ \times \mathbb{N})$, where $\mathcal{B}(\mathbb{R}_+ \times \mathbb{N})$ is the complete metric space of bounded functions (endowed with the sup metric), as:

$$Tf(z,t) = \max_{p,z'} \left\{ z - \frac{z'}{R_{t+1}} + \beta\alpha \left[\omega(p) + f(z'-p, t+1)\right] + \beta(1-\alpha) f(z', t+1) \right\},$$

subject to $z' \geq 0$, $z'/R_{t+1} - z \in \left[\underline{h}, \bar{h}\right]$ and $p \leq z'$. The mapping T satisfies the Blackwell's monotonicity and discounting sufficient conditions for a contraction with modulus β (Theorem 3.3 in Stokey and Lucas, 1989). From Banach fixed point theorem it admits a unique fixed point. By the Theorem of the Maximum (Theorem 3.6 in Stokey and Lucas) if f is continuous in z so it Tf since the constraint set,

$$\Gamma(z) \equiv \left\{(z', p) : z' \in \left[R_{t+1}\left(z + \underline{h}\right), R_{t+1}\left(z + \bar{h}\right)\right], 0 \leq \ p \leq z'\right\},$$

is a continuous correspondence. By the same reasoning Tf preserves the concavity of f with respect to z since the constraint set is a convex correspondence (see Theorem 4.7 in Stokey and Lucas, 1989). For any z such that $(z', p) \in \text{int}\Gamma(z)$ the differentiability of $W_t(z)$ follows from the Benveniste and Scheinkman theorem (Theorem 4.10 in Stokey and Lucas, 1989), i.e., $W_t'(z) = 1$. In order to establish the differentiability of W_t when (z', p) is on a boundary of $\Gamma(z)$ we apply results from Rincón-Zapatero and Santos (2009). We define the constraints on the control variables as:

$$\begin{aligned}
g^1(z, z', p) &\equiv z - \frac{z'}{R_{t+1}} + \bar{h} \geq 0 \\
g^2(z, z', p) &\equiv -z + \frac{z'}{R_{t+1}} - \underline{h} \geq 0 \\
g^3(z, z', p) &\equiv z' - p \geq 0.
\end{aligned}$$

Suppose g^1 is binding. Then the condition $D4$ (See Equation (4) in Rincón-Zapatero and Santos, 2009) requires that

$$\lim_{t\to\infty} \beta^t \prod_{s=0}^{t} R_{s+1} = 0.$$

■

Appendix A2: Map of equilibria

In the following we illustrate the set of equilibria for our economy for different values for $\bar{h}$, r, and α. We adopt the following functional forms: $u(y) = 2\sqrt{y}$ and $v(y) = y$. In Figure 12 we set $r = 0.04$. The colored area corresponds to equilibria with full depletion of real balances. The white area is when the equilibrium features partial depletion. Equilibria with full depletion exist when $\bar{h}$ is above a threshold. Moreover, as $\bar{h}$ increases N decreases since buyers can reach their targeted real balances in a smaller number of periods.

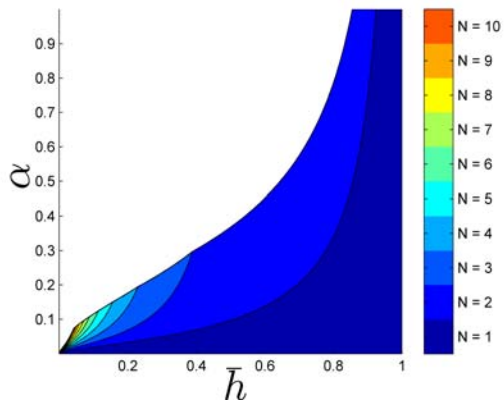

Figure 12: Typology of equilibria with full depletion in the $(\bar{h}, \alpha)$ space

In Figure 13 we allow r to vary and we set $\alpha = 0.1$. This figure is analogous to 5. For a given labor endowment N decreases as r decreases since more patient buyers have higher targeted real balances. When r is sufficiently low the equilibrium features partial depletion of real balances.

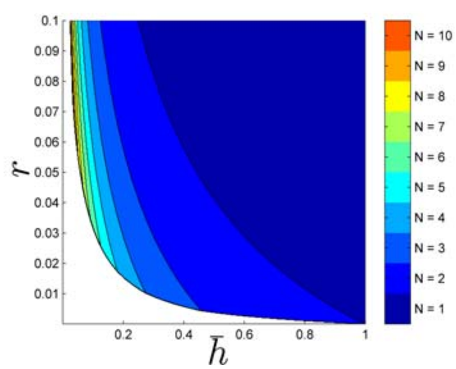

Figure 13: Typology of equilibria with full depletion in the $(\bar{h}, r)$ space

Appendix A3: Algorithm to compute value functions and stationary distributions

The algorithm to compute the value function, $W(z)$, and the distribution of real balances, F, at a steady-state equilibrium is composed of four steps:

Step 0. Fix some initial guess of $W_0(z)$ for the domain $z \in [0, \overline{z}]$, where $\overline{z}$ should be reasonably large. For example set $W_0(z)$ to be the closed-form case with $N = 2$, and $\overline{z} = 3(u')^{-1}(1 + r/\alpha)$.

Step 1. Iterate the following k-th value function

$$W_{k+1}(z) = z + \max_{\substack{h \le \overline{h}, \\ y \le h+z}} \{-h + \beta [\alpha [u(y) + W_k(z + h - y)] + (1 - \alpha) W_k(z + h)]\},$$

until it reaches some tolerance level, for example $\|W_{k+1}(z) - W_k(z)\| \le 10^{-6}$. Obtain the value function $W(z) = W_k(z)$.

Step 2. Obtain the policy function $h(z)$ and $y(z)$. Initiate $s = 1, 2...10^6$ sample points with $z_1^s = 0$.

Step 3. For each s, generate a uniform random variable U_s. Set $z_{t+1}^s = z_t^s + h(z_t^s) - \mathbb{I}_{U_s \le \alpha} p(z_t^s)$.

Step 4. Iterate z_t^s until $t = T = 10^5$. Obtain the stationary distribution $F(z)$ as the empirical CDF of z_T^s.

Appendix A4. Numerical Algorithm to Compute the Transitional Dynamics

Step 0. Fix a parameter that features a full-depletion stationary equilibrium with N mass points, for example we set $N = 4$ in the numerical example. Set a large $T > N$ number of periods of the transitional dynamics $\mathbf{R} = \{R_1, R_2, ...R_T\}$ and $\boldsymbol{\phi} = \{\phi_0, \phi_1, ...\phi_T\}$, for example we set $T = 100$ in the numerical example.

Step 1. Define the constraints: for all $t \geq N$

$$\phi_t \gamma M = \sum_{j=1}^{N-1} \mu_j \left[\sum_{n=1}^{j} \bar{h} \prod_{s=1}^{n} R_{t+1-s} \right] + \mu_N z_t^* \tag{94}$$

and for all $t \leq N- 1$

$$\phi_t \gamma M = \frac{\sum_{j=1}^{t-1} \mu_j \left[\sum_{n=1}^{j} \bar{h} \prod_{s=1}^{n} R_{t+1-s} \right] + \sum_{j=t}^{N-1} \mu_j \sum_{n=1}^{t} \bar{h} \prod_{s=1}^{n} R_{t+1-s} + \mu_N z_t^*}{1 - \frac{1}{\gamma} \sum_{j=t}^{N-1} \mu_j \left[m_j + (\gamma - 1) \right]} \tag{95}$$

where

$$m_j = \frac{\alpha(j-t)\bar{h}}{\bar{h} \left\{ 1 - (1-\alpha)^{N-1} \left[(N-1)\alpha + 1 \right] \right\} + \alpha (1-\alpha)^{N-1} z^*}.$$

The real balances of the N-1 th agent is

$$z_{N-1,t} = \begin{cases} (N-1)\, h R_0, \text{ if } t = 0 \\ h \left[(N-1-t) \prod_{s=1}^{t+1} R_{t+1-s} + \sum_{j=1}^{t} \prod_{s=1}^{j} R_{t+1-s} \right], \text{ if } t \in \{1...N-2\} \\ \sum_{n=1}^{N-1} h \prod_{s=1}^{n} R_{t+1-s}, \text{ otherwise} \end{cases}$$

The marginal value of real balances is

$$\lambda_t(0) = \sum_{j=1}^{N-1} \left\{ \mu_j \beta^j \omega' \left(\sum_{n=1}^{j} \bar{h} \prod_{s=1}^{n} R_{t+j-s+1} \right) \prod_{s=1}^{j} R_{t+j-s+1} \right\} + \beta^{N-1} \mu_N \prod_{j=1}^{N-1} R_{t+N-j}$$

The sufficient condition for full depletion is

$$1 + \frac{(1+r)/R_t - 1}{\alpha} \geq \lambda_t(0). \tag{96}$$

The sufficient condition for N mass point is

$$\omega'(z_{N-1,t}) > 1 + \frac{(1+r)/R_t - 1}{\alpha} \geq \omega'(z_{N,t}) \tag{97}$$

Step 2. Solve the standard constrained minimization problem: $\varepsilon \equiv \min_{\boldsymbol{\phi}, \mathbf{R}} (R_T - 1)^2$ subject to $R_t = \phi_t / \phi_{t-1}$, (94), (95), (96), and (97). Accept the transitional dynamics $\mathbf{R}$ if the error ε is less than some tolerance level, says 10^{-6}: after T period R_T is in the neighborhood of its steady state level 1. Notice that the algorithm does not always guarantee to reach a solution $\mathbf{R}$ for any parameter values, as the economy may feature partial depletion or the number of mass point changes along the transition dynamics.

A6. One-time, anticipated money injection

At the beginning of the CM in $t = 0$ agents are informed that the money supply will increase in the CM of $t = 1$ by $(\gamma - 1)M$, where $\gamma - 1$ is small. As in the previous section lump-sum money transfers are directed to buyers only. We focus on equilibria where the economy returns to its steady state one period after the money injection, in the CM of $t = 2$, and we solve the equilibrium by backward induction.

Aggregate real balances in $t = 2$ are at their steady-state level, $\gamma M \phi_2 = \alpha \bar{h} + (1-\alpha)z^*$, i.e., a measure α of buyers hold $\bar{h}$ real balances, which corresponds to their labor endowment, while a measure $1-\alpha$ hold their target, z^*. Aggregate real balances before the money injection are

$$\phi_1 M = \frac{\alpha \bar{h} + (1-\alpha)z^*}{\gamma R_2}, \tag{98}$$

where $R_2 = \phi_2/\phi_1 < 1$ solves an equation analogous to (45),

$$\frac{\alpha R_2 \bar{h} + (1-\alpha)z_2^*}{1 - \alpha\left(1 - \frac{1}{\gamma}\right)} = \alpha \bar{h} + (1-\alpha)z^*, \tag{99}$$

where z_2^* solves $\omega'(z_2^*) = 1 + (1 + r - R_2)/(\alpha R_2)$. In order to understand (99) recall that α agents enter period 2 with $R_2 \bar{h} + (\gamma - 1)\phi_2 M$ real balances while $1-\alpha$ agents enter with their targeted real balances, z_2^*. From (99) the rate of return of money, R_2, is unaffected by the announcement made in the initial period. The reason is that even though the announcement might affect the rate of return, R_1, and the distribution of real balances at the beginning of period 1, it does not affect the target for period 2.

Let us turn to the rate of return of money in period 1. Aggregate real balances in $t = 1$ are $\phi_1 M = \alpha R_1 \bar{h} + (1-\alpha)z_1^*$. From (98) R_1 solves

$$\alpha R_1 \bar{h} + (1-\alpha)z_1^* = \frac{\alpha \bar{h} + (1-\alpha)z^*}{\gamma R_2}. \tag{100}$$

The left side of (100) is increasing in R_1 and it is greater than the right side when $R_1 = 1$ if $\gamma R_2 > 1$. Hence, if $R_2 > \gamma^{-1}$ then $R_1 < 1$, i.e., $\phi_0 > \phi_1$. The value of money declines over time until it reaches its new steady-state value in the CM of $t = 2$. Because $R_1 < 1$ the real balances of all buyers in the DM of $t = 1$ are lower. Hence, aggregate DM output falls in $t = 1$, before the money injection. It increases in $t = 2$ following the money injection if $v'' > 0$ because of the mean-preserving redistribution of real balances. In the left panel of Figure 14 we represent the trajectories for aggregate DM output and the value of money.

Consider next the case where $\gamma R_2 < 1$ which implies $R_1 > 1$. The path for the value of money is such that $\phi_0 < \phi_1$ and $\phi_1 > \phi_2$. The value of money increases first and then decreases. In the DM of $t = 1$ all consumption levels increase whereas in the DM of $t = 2$ the consumption of the richest decreases while the consumption of the poorest increases. Welfare increases in both periods. In the right panel of Figure 14 we represent the trajectories for aggregate DM output and the value of money. Finally, in the knife-edge case where $\gamma R_2 = 1$ then $R_1 = 1$ so that prices stay constant between $t = 0$ and $t = 1$. The output levels in the DM of $t = 1$ are unchanged.

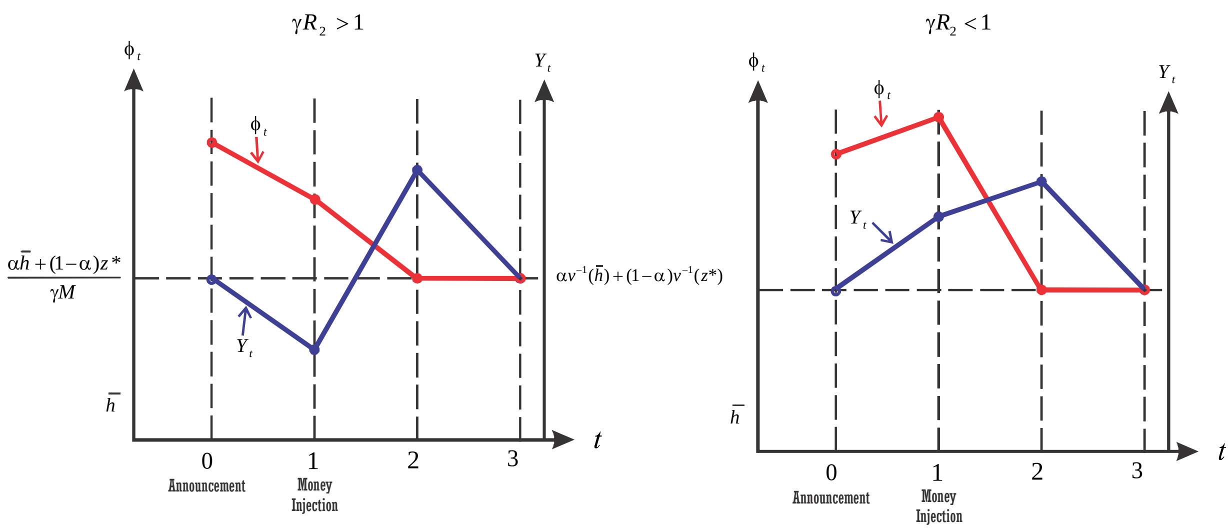

Figure 14: Small, anticipated, money injection