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Abstract

When society acknowledges the potential for individual priors to be misspecified, it encounters a *structured* ambiguity in the process of making collective decisions through aggregation. This paper highlights various principles that provide a means to overcome this structured ambiguity by leveraging the consensual core of individual priors. This is achieved through a *sequential* aggregation mechanism, wherein the decision-making process is divided into multiple stages. Aggregation is thus conducted progressively. We demonstrate that, in comparison to conventional synchronized aggregation, this approach significantly broadens the scope of feasible aggregations.

JEL classification: D7, D8.

Keywords: Structured Ambiguity, Misspecification, Sequential Aggregation, α -Maxmin Expected Utility

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1 INTRODUCTION

Society often faces the challenge of making complex economic decisions, such as those related to environmental economic policy or government debt levels. Studies related to these decisions often encounter difficulty in reaching agreement. There are several reasons for this lack of consensus. Firstly, these studies often employ different methodologies, leading to varying probability estimates. Secondly, due to the complexity of the situations, the models used for evaluation may be misspecified ([Hansen and Sargent \(2022\)](#)). As a result, standard approaches like expected utility theory, which relies on belief aggregation methods ([Gilboa, Samet and Schmeidler \(2004\)](#)), and cost-benefit analysis do not effectively evaluate the decisions to be made. This underscores the need for alternative approaches that can address the complexities and misspecification associated with these economic decisions.

To illustrate this point with a simple example, consider a city that needs to propose new environmental standards for future constructions within its jurisdiction. Suppose the best estimate for temperature variations indicates that the average temperature increase in the city will not exceed two degrees. In this scenario, adopting standard green building practices would be a reasonable decision. However, if the average temperature increase reaches or exceeds five degrees, a more stringent set of standards would be appropriate. Therefore, the predicted outcomes differ significantly depending on whether the chosen standard addresses a temperature increase of 2°C or 5°C . To make the best possible decision, policymakers need to consider the entire range of potential outcomes by examining the distribution of temperature changes to evaluate the robustness of the decision. Thus, in this example, the final decision is heavily dependent on the evaluation of climate sensitivity.

However, due to the heterogeneity of climate models, statistical methods, and observed data used by scientists, the evaluation of climate sensitivity is highly inconsistent ([Meinshausen et al. \(2009\)](#)). Since the different estimates are not independent from each other ([Tebaldi and Knutti](#)

(2007)), "we cannot objectively single out a *best* study" (Millner, Dietz and Heal (2013), p. 24). Furthermore, these models themselves have significant flaws, and the possibility of misspecification cannot be dismissed.

Continuing with the example, assume there are four possible temperature changes: $+1^\circ\text{C}$, $+2^\circ\text{C}$, $+5^\circ\text{C}$, $+10^\circ\text{C}$. We also assume that there are two models evaluating these possibilities. Model 1 provides the estimates $p_1 = (0.1, 0.2, 0.5, 0.1)$, while Model 2 gives the estimates $p_2 = (0.1, 0.5, 0.2, 0.1)$. It is important to note that both models agree on the probability assigned to a 1°C and a 10°C temperature increase, each being 0.1. Therefore, the analyst should adhere to this consensus, that is, $\pi(1) = \pi(10) = 0.1$. However, the estimates for a 2°C and a 5°C temperature change differ significantly between the two models, precluding a definitive judgment by the analyst. At the same time, the analyst recognizes the significant limitations of the models and acknowledges that both models carry a non-negligible possibility of making incorrect evaluations. Based on this, the analyst retains multiple probabilistic possibilities for 2° and 5° , encompassing all probability extensions consistent with $\pi(1) = \pi(10) = 0.1$, namely, $\mathbb{P}_\pi = \{p : p(1) = p(10) = 0.1\}$. This approach includes not only the information directly reflected by the models but also captures the missing information that the models do not possess. Therefore, one approach proposed herein is the α -maxmin expected utility theory (Gilboa and Schmeidler (1989); Ghirardato, Maccheroni and Marinacci (2004); Gul and Pesendorfer (2015)). This approach involves considering a set of prior probabilities that includes all the probability extensions based on the agreed probabilities.

To begin, it is necessary to clarify what we mean by *ambiguity*. Here, it is assumed that each individual has their own statistical model based, among other things, on a probability distribution, and society builds its own model by aggregating these different individual models. However, individual probabilistic models are not necessarily consistent with each other and can even vary considerably. Additionally, individuals may lack full confidence in the probability distribution they use to represent their beliefs. In reality, the process of model construction is inherently a simplification

and approximation of the actual situation. Consequently, as Hansen (2014) points out, the possibility of model misspecification can never be entirely ruled out. Moreover, the model validation process often suffers from a lack of sufficiently relevant data, complicating the accurate identification of the model itself. This is precisely what Manski, Sanstad and DeCanio (2021) refers to as *partial identification*, which explains why the corresponding uncertainty is termed *deep uncertainty* or *structured uncertainty*. In this paper, the ambiguity we are discussing is clearly of this nature, which is why we refer to it as *structured ambiguity*.

Assume now that each individual model corresponds to the Expected Utility (EU) framework. Consequently, the social framework has to manage the structured ambiguity generated by the aggregation of these individual EU models (see, Epstein and Zhang (2001)). In the context of α -maxmin EU framework, society constructs its model by determining its set of probabilities, choosing its attitude towards structured ambiguity, namely, the value α , and ultimately its social utility. As this construction mechanism proceeds by aggregation, it must satisfy certain probing constraints.

To ensure that social beliefs are robust to misspecification, we first identify a set of events *approved* by all individuals in the sense that they have the same prior regarding those events, and we consider the probability distribution p over this set. Then, we introduce *opinions*, one called *cautious* and the other *bold*, that must align with society's ranking on lotteries defined on approved events. The relationship between social preferences and opinions is determined through two axioms, Unanimity and Independence. So, while social preferences can be represented by an α -maxmin expected utility function over opinions, it is shown that the set of social beliefs robust to misspecification corresponds to the set of extensions of p to all events.

The new approach to aggregation we propose can be broken down into several steps. The first step is to apply a Pareto condition *à la* Harsanyi linking individual and social preferences. As the principle only applies to acts defined as *approved acts*, i.e. lotteries on approved events, this ensures that the social probabilities are collectively consistent, while the social utility

function corresponds to the weighted average of individual utilities. Due to its lack of confidence in individual beliefs, society completely ignores individuals' perceptions and evaluations of ambiguous events. Instead, it regards approved events as an approximation and estimate of ambiguous events. This illustrates the very principle of the aggregation mechanism we propose, which is based on an *objective* uncertainty that is shared by all to gradually introduce a *subjective* uncertainty through the influence of opinions on social preferences. It is important to note that the derived social preferences are actually incomplete. It turns out that these preferences can be represented by an expected utility in the Bewley sense ([Bewley \(2002\)](#)), where beliefs belong to a certain subset of probability expansions. Undoubtedly, the aggregation of these newly derived preferences, once again applying a principle of unanimity, results in the set of probability expansions associated with α -maximal social preferences comprising a subset of all probability expansions.

With the exception of a few rare contributions ([Pivato \(2022\)](#)), theories of aggregation under uncertainty generally satisfy the assumption of methodological individualism. In other words, social values are usually determined by individual values ([Harsanyi \(1955\)](#); [Gilboa, Samet and Schmeidler \(2004\)](#); [Billot and Qu \(2021\)](#); [Dietrich \(2021\)](#)), and the range of social beliefs is often bounded by the set of individual beliefs ([Crès, Gilboa and Vieille \(2011\)](#); [Alon and Gayer \(2016\)](#); [Qu \(2017\)](#); [Danan et al. \(2016\)](#)). Unlike these previous works, this article adheres solely to a principle of partial individualism. It only uses probabilities for which individuals are collectively consistent, while retaining the other event probabilities that incorporate divergent opinions. In situations of structured ambiguity, such as environmental, fiscal, or medical issues, individual models often lack sufficient theoretical and empirical evidence to be convincing and are therefore prone to judgment errors. The possibility of individual models being misspecified should encourage society to be more cautious in choosing which model to use. Therefore, the main strength of this article is to provide a novel and reasonable method for society to aggregate individ-

ual preferences when the models employed by individuals are likely to be misspecified. Recently, (Bommier et al., 2021) introduced the concept of *distribution-consensus*, which entails unanimous agreement among individuals on expected utility and a consensus regarding the distribution of outcomes. It is worth noting that their framework differs from ours, and their distribution averaging model is significantly distinct from our α -maxmin model.

An important innovation of this article is the sequential aggregation mechanism. Indeed, there don't seem to be any other papers using such a mechanism. Standard aggregation is synchronized, meaning it only involves a single step. It is well-known that synchronized aggregation often encounters difficulties that affect the aggregation result. For example, Mongin (1995) and Mongin and Pivato (2020) highlight that unanimity in synchronized aggregation can be *spurious*. On the other hand, in the case of sequential aggregation, it is possible to decompose the problem into multiple steps and sequentially aggregate the parameters we need. Compared to synchronized aggregation, sequential aggregation significantly increases the possibilities of non-dictatorial aggregation. In practice, social decisions often result from a process of repeated refinement. In this sense, sequential aggregation provides a more accurate description of the actual process of social decision-making.

This paper is organized as follows. Section 2 contains the framework and aggregation result for the social utility function. Section 3 formally outlines and investigates the sequential aggregation method and presents the main result. Section 4 considers some extensions of this result. We conclude in Section 5. All proofs are in the Appendix.

2 THE MODEL

Let (S, Σ) denote a σ -measurable space, where S is a set of *states of nature* and Σ is a σ -algebra of *events*. Let X a set of *outcomes*, which is assumed to be a connected and compact metric space. A social *act* is a Σ -measurable

simple function $f : S \rightarrow X$ and $\mathcal{F}$ is the set of all social acts.¹ An act f is *constant* if there is $x \in X$ such that $f(s) = x$, for all $s \in S$. Depending on the context, we may abuse the notation and refer $f(s)$ or x as a constant act.

Society is a set of individuals $\mathcal{I} = \{1, \dots, n\}$. Each individual $i \in \mathcal{I}$ has preferences over $\mathcal{F} \times \mathcal{F}$, that is a binary relation $\succsim_i \subset \mathcal{F} \times \mathcal{F}$. Social preferences are denoted by $\succsim \subset \mathcal{F} \times \mathcal{F}$. A function $V : \mathcal{F} \rightarrow \mathbb{R}$ represents preferences $\succsim$ on $\mathcal{F}$ if, for all $f, g \in \mathcal{F}$, $f \succsim g$ if and only if $V(f) \geq V(g)$.

Definition 1. A function $V : \mathcal{F} \rightarrow \mathbb{R}$ is a *subjective expected utility* function if there exists a unique probability measure π on Σ , and a utility function u on X , such that, for $f \in \mathcal{F}$:

$$(1) \quad V(f) = \int_S u(f) d\pi.$$

Assumption 1 — (SEU individuals). For all $i \in \mathcal{I}$, individual preferences $\succsim_i$ are represented by a subjective expected utility (SEU). This representation corresponds to the unique pair (u_i, π_i) as in (1) (up to the affine transformation of u_i), where u_i is continuous and non-constant and π_i is countably additive and nonatomic.

Assumption 1 is based on Savage's postulates, which mandate that all individual preferences be represented by a SEU. However, the model does not impose that social preferences admit a SEU representation in the presence of heterogeneous individual beliefs. This sets it apart from [Gilboa, Samet and Schmeidler \(2004\)](#) and [Mongin \(1995\)](#).

[Harsanyi \(1955\)](#) assumes that under objective uncertainty, when all individuals share the same beliefs about all events, social preferences are represented by an expected utility function. However, conflicts may arise when individual beliefs differ, and in such cases, society must compromise with these heterogeneous beliefs. Therefore, as noted by [Diamond \(1967\)](#), it is neither intuitive nor reasonable to determine the form of preference repre-

¹The *topology of pointwise convergence* on $\mathcal{F}$ is defined as the relative topology with respect to the product topology on X^S .

sensation before establishing the rule for aggregating beliefs. For this reason, it is more appropriate to assume that social preferences admit a SEU representation for approved acts, while remaining *representation agnostic* for other acts.

Definition 2. An event A is *approved* if $\pi_i(A) = \pi_j(A)$, for all $i, j \in \mathcal{I}$.

Let $\mathcal{A}$ denote the collection of all approved events, such that an event A belongs to $\mathcal{A}$ if all individuals agree on its probability. An act f is categorized as a *lottery*, what we refer to as an *approved act* in the introduction, if every measurable subset of outcomes Y is approved, that is $f^{-1}(Y) \in \mathcal{A}$, for all $Y \subset X$. We use $\mathcal{L}$ to symbolize the set of lotteries. If all individuals agree on the probability of each event, then $\mathcal{A} = \Sigma$. However, if there is a belief heterogeneity, then the set of approved events is only a subset of Σ , and in general, does not form an algebra. Recall that a λ -system $\Lambda \subseteq 2^\Omega$ is a collection of subsets such that:

- (i) $\Omega \in \Lambda$,
- (ii) if $E \in \Lambda$, then $E^c \in \Lambda$,
- (iii) for any countable collection of disjoint events $E_k \in \Lambda$, $\bigcup_k E_k \in \Lambda$.

In fact, the collection of all the approved events consists a λ -system.²

Lemma 1. The collection $\mathcal{A}$ of approved events is a λ -system.

We say that $\pi : \mathcal{A} \rightarrow [0, 1]$ is a probability measure on approved events if:

- (i) $\pi(\emptyset) = 0 \leq \pi(A) \leq \pi(S) = 1$, for every $A \in \mathcal{A}$, and
- (ii) $\pi(\bigcup_k A_k) = \sum_k \pi(A_k)$, for any countable collection of disjoint events $A_k \in \mathcal{A}$.

It is worth noting that any probability measure defined on Σ and restricted to $\mathcal{A}$ is also a probability measure on $\mathcal{A}$. Therefore, the probability measure π , which is defined on the approved events and satisfies

²Border, Ghirardato and Segal (2008) has demonstrated that the approved set $\mathcal{A}$ is also a σ -algebra. Given that this observation does not impact any of our results, we have decided to maintain our findings as originally presented.

$\pi(A) = \pi_i(A)$, for all $A \in \mathcal{A}$, all $i \in \mathcal{I}$, can be considered as a probability measure on $\mathcal{A}$.

Definition 3. Given a collection of approved events $\mathcal{A}$, a function $V : \mathcal{F} \rightarrow \mathbb{R}$ is a *restricted* SEU (RSEU) with respect to $\mathcal{A}$ if there exist a probability measure π on $\mathcal{A}$ and a continuous and nonconstant utility function u on X , such that, for $f \in \mathcal{L}$:

$$(2) \quad V(f) = \int_S u(f) d\pi.$$

Assumption 2 — (RSEU society). Social preferences $\succeq$ are represented by a RSEU V . This representation corresponds to the unique triplet $(\mathcal{A}, \pi, u)$ as in (2) (up to the affine transformation of u) where the function u is continuous and non-constant. Additionally, we assume that V is both monotonic³ and continuous (in the topology of pointwise convergence).

Thus, according to Assumption 2, social preferences $\succeq$ must conform to Savage's postulates, which are confined to the set of lotteries, i.e. the set of approved acts. For a detailed discussion of RSEU, we refer to [Epstein and Zhang \(2001\)](#) and [Kopylov \(2007\)](#) for the formal characterization.

The first principle proposed in this paper is the Pareto principle, but it exclusively applies to preferences concerning lotteries.

Harsanyi Pareto condition (HPC). For every lotteries $f, g \in \mathcal{L}$, if $f \succeq_i g$, for all $i \in \mathcal{I}$, then $f \succeq g$.

HPC states that if every individual favors the first lottery over the second, then society would also prefer the first lottery. According to individual beliefs, each lottery corresponds to an identical von Neumann-Morgenstern lottery. As a result, HPC can be viewed as a natural extension of the objective uncertainty condition to the subjective uncertainty.

³ V is *monotonic* if $u(f(s)) \geq u(g(s))$, for all $s \in S$, implies $V(f) \geq V(g)$.

Definition 4. Given $\{(u_i, \pi_i)\}_{i \in \mathcal{I}}$, a RSEU $(\mathcal{A}, u, \pi)$ is *collectively consistent* if $\mathcal{A}$ is the set of approved events with $\pi(A) = \pi_i(A)$, for all $i \in \mathcal{I}$ and all $A \in \mathcal{A}$, and it is *utilitarian* if u is a convex combination of $\{u_i\}_{i \in \mathcal{I}}$. Moreover, a RSEU is said to be *consistently utilitarian* if it is both collectively consistent and utilitarian.

The next result characterizes the relation between HPC and a consistently utilitarian RSEU.

Theorem 1. HPC holds if and only if social preferences $\succeq$ are represented by a consistently utilitarian RSEU.

This theorem does not prescribe a particular form of representation for all acts in general. Instead, it characterizes social utility as a convex combination of individual utilities and social beliefs when they are based on approved events. As HPC only pertains to social preferences for approved acts, this result can be considered as an extension, albeit rather limited, of Harsanyi's aggregation theorem from objective uncertainty to subjective uncertainty.

3 SOCIAL OPINIONS

Since all individuals share the same beliefs about each approved act, i.e. each lottery, social preferences regarding lotteries based on HPC can be seen as a consensus evaluation of these lotteries. It is important to note that society only agrees on approved events and therefore does not pass judgment on events with heterogeneous estimates.⁴ Social preferences restricted to approved acts are unanimously accepted by all individuals. Consequently, these preferences naturally serve as a *reference* for society when it seeks to evaluate other acts that are not lotteries.

Formally, social preferences $\succeq$ are said to *match* with preferences $\succeq^*$ on $\mathcal{F}$ if $\succeq$ agrees with $\succeq^*$ on lotteries, i.e. $f \succeq g$ if and only if $f \succeq^* g$, for all $f, g \in \mathcal{L}$.

⁴For convenience, these events can be referred to as *ambiguous*.

Thus, if social preferences $\succeq$ match with $\succeq^*$, then $(\mathcal{A}, \pi, u)$ represents $\succeq^*$ restricted to $\mathcal{L}$. We say preferences are defining a *social opinion* if they match with $\succeq$ and if they admit a RSEU representation. Let Θ be a collection of *under-consideration social opinions*. For any opinion $\theta \in \Theta$, $\succeq_\theta$ denotes the corresponding social preferences for the opinion θ . The social opinions have an identical ranking, concerning approved acts. However, they differ not only in their estimates of socially ambiguous events but also in their attitudes towards these events.

For instance, a consistent SEU is a social opinion. Given $(\mathcal{A}, u, \pi)$, let $\mathbb{P}_\pi$ be the set of all the *extensions* of π on Σ . We say a social opinion θ is *probabilistic* if there is $p \in \mathbb{P}_\pi$ such that:

$$f \succeq_\theta g \Leftrightarrow \int u(f)dp \geq \int u(g)dp.$$

In this case, we can write $\theta = p$. In a similar manner, a social opinion can be *multiple prior-EU* (MEU) in the sense of [Gilboa and Schmeidler \(1989\)](#).

Society faces several opinions that are both rational and mutually contradictory. We argue here that society should respect the following two principles if it intends to formalize its preferences.

Unanimity. For all acts $f, g \in \mathcal{F}$, if $f \succeq_\theta g$, for all $\theta \in \Theta$, then $f \succeq g$.

The principle of unanimity states that if all social opinions prefer act f over act g , then society as a whole also prefers act f . This principle is particularly compelling because the consensus among social opinions on the ranking of outcomes prevents any spurious unanimity, as described by [Mongin \(1995\)](#), from occurring (see the introduction of this paper for an illustration of such spurious unanimity).

For $x, y \in X$ and event A , a binary act, written xAy , describes an act such that the outcome is x if event A is realized and y otherwise.

Independence. For all acts $f, g \in \mathcal{F}$, $x, y \in X$ and $A \in \mathcal{A}$, if for every $\theta \in \Theta$, there exists $z_\theta \in X$ such that $f \sim_\theta z_\theta$ and $g \sim_\theta z_\theta Ay$, then $f \sim x$ implies $g \sim xAy$.



Figure 1: Preference Aggregation Process

The principle of independence states that if, for all social opinions, an act g is indifferent to a binary lottery, whose outcomes consist of the corresponding certainty equivalent of f and y , then society also considers g to be indifferent to the binary act whose outcomes are certainty equivalent of f and y . It is worth emphasizing that this axiom is intuitive. In the context where randomization is allowed, it simply states that if g is indifferent to a mixture of f and constant y for all social opinions, then society will also be indifferent. However, it is important to note that we are exclusively considering the Savage setting and excluding the possibility of a random device. In this context, the axiom serves as a re-expression of the classic one into the Savage setting. (Our proposed aggregation process is summarized above in Figure 1.)

Note that in the case of probabilistic social opinions, i.e. when $\Theta = \mathbb{P}_\pi$, it becomes necessary to apply a probabilistic principle of unanimity of the following form: “for any $f, g \in \mathcal{F}$, if $f \succeq_\theta g$, for any $\theta \in \mathbb{P}_\pi$, then $f \succeq g$ ”. However, we can immediately see that this puts almost no restrictions on social preferences, which limits the interest of the process we propose. Therefore, it is natural to consider a subset of all the probabilistic social opinions.

3.1 Pole Opinions

In an environment and era where societies are constantly inundated with information primarily disseminated through social media, it is increasingly difficult for individuals to make independent decisions separate from the expressed opinions. It is therefore both prudent and effective to focus on the most polarized opinions, those based on an extreme apprehension of the environment, which thus favor the consideration of the best and worst possible scenarios. We distinguish two main opinions, one that we call *cau-*

tious and the other bold.

Definition 5. For any act $f \in \mathcal{F}$ and $x \in X$, $f \succ_{\text{caut}} x$ iff there is a lottery $g \in \mathcal{L}$ such that $f(s) \succeq g(s)$, for all s and $g \succ x$. The relation $\succeq_{\text{caut}}$ is referred to as a *cautious* social opinion.

Cautious social opinion is characterized by a cautious way of valuing each option f .⁵ Here, the expected utility of an option f is achieved by a sort of approximation-from-below-process such that it is equal to the *highest* expected utility associated to the f -dominated lotteries. As an illustration, let us consider the case of a comparison between an option f and a constant act x . The cautious social opinion does not allow a direct comparison of f with x . Instead, it proposes to make only an indirect comparison through lotteries that are dominated by f . Since monotonicity immediately implies that f is preferred to all lotteries that it dominates, it follows that, if any dominated lottery is preferred to x , then, by transitivity, the option f is also preferred to x .

Definition 6. For any act $f \in \mathcal{F}$ and $x \in X$, $x \succ_{\text{bold}} f$ iff there is a lottery $g \in \mathcal{L}$ such that $g(s) \succeq f(s)$, for all s and $x \succ g$. The relation $\succeq_{\text{bold}}$ is referred to as a *bold* social opinion.

Similarly, a bold social opinion cannot directly compare option f with constant act x . It also applies an indirect comparison. However, a bold opinion uses the dominant lotteries to evaluate f in a reckless way this time. If x is preferred to any dominant lottery, then x is also preferred to option f . According to this view, the expected utility of an option is equal, through an approximation-from-above-process, to the *lowest* expected utility associated to the dominant lotteries.

Why should a society give special consideration to cautious and bold opinions? First, both opinions respect the monotonicity and transitivity of preferences, which ensures that they are indeed rational in the theoretical

⁵In fact, cautious opinion is similar to the conditions of Consistency and Caution in Gilboa et al. (2010).

sense. Second, these two opinions together determine the lower and upper bounds between which the utility of an option can vary. The cautious opinion establishes the lower value and the bold opinion the upper value. Taking these two opinions into account helps to avoid overestimating or underestimating the utility of an option. Finally, and most importantly, the difference between the two values of an option's expected utility can serve as a measure of the intensity of the conflict of opinion. Therefore, it can be expected that society will seek to manipulate this gap in order to strategically use the extent of the conflict of opinion.

Let $\mathbb{P}_\pi$ be the collection of all probability extensions of a probability π . It is easy to see that, for the cautious opinion, if $f \succ_{\text{caut}} x$, then the expected utility of f is greater than the expected utility of x , for all extensions of π . Therefore, we can also interpret the cautious opinion such that an option f is preferred to a constant act x only if its expected utility is greater than the expected utility of x , for every possible extension of π . Similarly, for the bold opinion, if $x \succ_{\text{bold}} f$, then the expected utility of f is smaller than the expected utility of x , for every possible extension of π .

Given a subset $\mathbb{P} \subseteq \mathbb{P}_\pi$ of probability extensions of π , we can now define the notion of generalized maxmin expected utility.

Definition 7. A function $V : \mathcal{F} \rightarrow \mathbb{R}$ is a *generalized maxmin expected utility* (GMEU) if there exist a nonempty compact and convex set $\mathbb{P}$ of probabilities on Σ , a utility function u on X and a monotonic function $W : u(X) \times u(X) \rightarrow \mathbb{R}$ with $V(x) = W(u(x), u(x))$, for all $x \in X$, such that:

$$(3) \quad V(f) = W(\bar{u}_{\mathbb{P}}^f, \underline{u}_{\mathbb{P}}^f),$$

where

$$\bar{u}_{\mathbb{P}}^f \equiv \max_{p \in \mathbb{P}} \int u(f) dp \quad \text{and} \quad \underline{u}_{\mathbb{P}}^f \equiv \min_{p \in \mathbb{P}} \int u(f) dp.$$

Given $(\mathcal{A}, u, \pi)$, a GMEU function V is a *generalized Hurwicz expected utility* (GHEU) if $\mathbb{P} = \mathbb{P}_\pi$.

Note that the definition of a GMEU depends on *polar* expected utilities, in the sense that the utility of an act f is determined only by its highest possible expected utility $\bar{u}_{\mathbb{P}}^f$ and its lowest possible expected utility $\underline{u}_{\mathbb{P}}^f$. A particular occurrence of GMEU is the case of α -MEU utilities:⁶

Definition 8. A function $V : \mathcal{F} \rightarrow \mathbb{R}$ is a α -*maxmin expected utility* (α -MEU) if there exist a nonempty compact and convex set $\mathbb{P}$ of probabilities on Σ and a utility function u on X such that:

$$(4) \quad V(f) = \alpha \max_{p \in \mathbb{P}} \int u(f) dp + (1 - \alpha) \min_{p \in \mathbb{P}} \int u(f) dp,$$

where $\alpha \in [0, 1]$.

Given $(\mathcal{A}, u, \pi)$, an α -MEU function V is a *Hurwicz expected utility* (HEU) if $\mathbb{P} = \mathbb{P}_{\pi}$. The α -MEU case includes standard MEU and maxmaxEU as special cases where $\alpha = 0$ and $\alpha = 1$, respectively.

The following theorem shows that if society is only sensitive to cautious and bold opinions, then the satisfaction of Unanimity is equivalent to the existence of uniformly utilitarian GHEU social preferences.

Theorem 2. Suppose $\Theta = \{\text{caut}, \text{bold}\}$. Unanimity holds if and only if social preferences $\succeq$ are represented by a consistently utilitarian GHEU.

A special case of Theorem 2 concerns a society that is sensitive to only one opinion. For example, if social sensitivity were cautious, then social preferences would be represented by a standard MEU.

Corollary 1. Social preferences are cautious, i.e. $\succeq = \succeq_{\text{caut}}$, iff they are represented by a consistently utilitarian MEU with $\mathbb{P} = \mathbb{P}_{\pi}$. In the same way, social preferences are bold, i.e. $\succeq = \succeq_{\text{bold}}$, iff they are represented by a consistently utilitarian maxmaxEU with $\mathbb{P} = \mathbb{P}_{\pi}$.

⁶An axiomatization result in the Anscombe-Aumann setting can be found in [Kopylov \(2003\)](#).

(This result follows immediately from the proof of Theorem 2. Therefore, we omit the proof.) Corollary 1 shows that cautious and bold opinions are based on the same underlying beliefs, which correspond to $\mathbb{P}_\pi$. In reality, they only differ in their system of evaluating ambiguous acts, i.e. acts other than lotteries. Unanimity requires society to consider only two opinions when evaluating acts, but it does not specify a precise functional form for expected utility. This gives society some flexibility in choosing the function that represents its preferences in a complex environment, and it allows society to remain undecided about how it intends to make its decisions. To derive a more concrete form of representation, such as α -MEU, Independence is then necessary.

Thus, Theorem 3 below provides an axiomatic characterization of social preferences of type HEU. When society respects both Unanimity and Independence, then social preferences admit a representation of type α -MEU, where social utility is defined as a convex combination of individual utilities and social beliefs as any extension of the probability π .

Theorem 3. *Suppose $\Theta = \{\text{caut}, \text{bold}\}$. Unanimity and Independence hold if and only if social preferences $\succeq$ are represented by a consistently utilitarian HEU.*

This result characterizes a society that constructs its preferences based on a weighted average of cautious and bold opinions. The set of socially ambiguous events coincides with those shared by the two opinions. Thus, society considers that the evaluation of the likelihood of an ambiguous event A should not consist of a single number, but rather an interval of numbers. Roughly speaking, the lower and upper bounds of this interval are measured by the probability of the largest approved event contained in A and the probability of the smallest approved event containing A , respectively. It can be said that the social approach to evaluating the likelihood of socially ambiguous events is, in a way, identical to the approach proposed by Dempster-Shafer for dealing with individual uncertainty.

Given that ambiguity aversion is a widely observed phenomenon, one may question the relevance of bold opinion in the construction of social

preferences. It is entirely conceivable that cautious opinion is more common and that bold opinion is indeed rarer. However, in a collective decision-making framework, as emphasized by Will Durant, it is often the case that "a united minority acting against a divided majority" can have a significant influence. Therefore, it is legitimate to consider that an HEU-type society is descriptively more relevant than a society that would have been a priori described as MEU-type.

3.2 Opinion Refinement

The cautious and bold opinions may appear too extreme. For an ambiguous event, the cautious social estimate is actually lower than the lowest individual cautious estimate. Similarly, the bold social estimate is higher than the highest individual bold estimate. In this subsection, we aim to identify a possible way to refine the cautious and bold opinions to avoid an overly radical social estimate of these ambiguous events.

We know that the cautious and bold opinions draw from the same set of social beliefs, which is the set of all probability extensions of π . Therefore, it is likely that the two opinions share the same *robust preference relation*, as defined below.

Definition 9. For $f, g, h \in \mathcal{F}$ and $A \in \mathcal{A}$, h is A -mixture of f, g if, for all s , $h(s) \sim f(s)Ag(s)$.

We denote A -mixture of f and g by $f[A]g$.

Remark. For any preferences $\succsim^*$ on $\mathcal{F}$ matching with $\succsim$, the A -mixture of f and g with respect to $\succsim^*$ is the same as the one with respect to $\succsim$.

Definition 10. Given a social opinion $\theta \in \Theta$ and two acts $f, g \in \mathcal{F}$, f is said to be *robustly preferred* to g , denoted $f \succeq_\theta g$, iff, for any $h \in \mathcal{F}$:

$$f \succeq_\theta g \iff f[A]h \succeq_\theta g[A]h \quad \text{for all } A \in \mathcal{A}.$$

An act f is robustly preferred to another act g if all event-mixtures of f and g with another act h reproduce between them the same ranking. We note that the social opinion $\succeq_\theta$, for $\theta \in \{\text{caut}, \text{bold}\}$, restricted to lotteries, i.e. approved acts, is a robust preference. However, in general, $\succeq_\theta$ is defined on $\mathcal{F}$ and not intended to be restricted to lotteries only. For example, if one option state-wisely dominates another option, it is obvious that the former will be robustly preferred to the latter. Similarly, society robustly prefers f to g , written $f \succeq g$, if $f[A]h \succeq g[A]h$, for all $h \in \mathcal{F}$ and $A \in \mathcal{A}$. Since cautious and bold opinions share the same beliefs, it is imperative to require that if they both believe that one act is robustly preferable to another, so does society:

Robust unanimity. If $f \succeq_\theta g$, for all $\theta \in \{\text{caut}, \text{bold}\}$, then $f \succeq g$.

What does robust unanimity imply? First, robust preferences are a priori incomplete. Therefore, social robust preferences $\succeq$ are Bewley preferences: there exists a subset $\mathbb{P}$ of $\mathbb{P}_\pi$ such that one act is robustly preferred to another if and only if the expected utility of the former is greater than that of the latter, this for each probability in $\mathbb{P}$. If society respects robust unanimity, then social robust preferences can be interpreted as a reference point for the refinement of social opinions.

Definition 11. A social opinion is *pessimistic*, i.e. $\theta = \text{pessi}$, if $f \not\succeq x$ implies $x \succeq_{\text{pessi}} f$.

A pessimistic opinion holds that a constant act x is always better than any act f whenever f is not robustly preferred to x . In other words, when there is an event A such that an A -mixture with x is better than the same mixture with f , then the pessimistic opinion believes that x is a better act.

Definition 12. A social opinion is *optimistic*, i.e. $\theta = \text{opti}$, if $x \not\succeq f$ implies $f \succeq_{\text{opti}} x$

An optimistic opinion believes that any act f is better than a constant act x if x is not robustly preferred to f . In other words, whenever there is

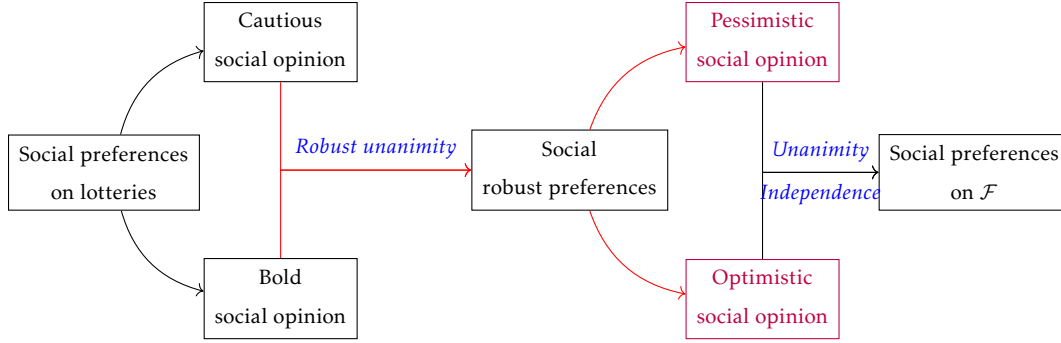


Figure 2: Opinion Process

an event whose associated mixture with f is better than the same mixture with x , then the optimistic opinion believes that f is a better act.

See Figure 2 above for the formation of pessimistic and optimistic opinions.

We want to emphasize here that both pessimistic and optimistic opinions depend on social robust preferences $\succeq$. Robust unanimity requires only that $\succeq$ be Bewley-type preferences characterized by $\mathbb{P} \subseteq \mathbb{P}_\pi$. Therefore, $\mathbb{P}$ may as well contain all probability extensions of π or be a singleton. If $\mathbb{P} = \mathbb{P}_\pi$, then the pessimistic and optimistic opinions are absolutely identical to the cautious and bold opinions, respectively.⁷ If $\mathbb{P}$ is a singleton, then the pessimistic and optimistic opinions coincide and constitute an opinion that can be represented by a SEU, i.e. what we will call later a *Bayesian* opinion.

In this process, society takes the initiative to influence social opinions. If the heterogeneity of individual estimates of an ambiguous event is low enough, society may agree to treat it as an approved event by assigning it a number. As a result, social opinions will not conflict with respect to the estimation of this event. Similarly, society can use the minimum and maximum individual probabilities to estimate each ambiguous event, which will then lead to define $\mathbb{P}$ as the convex hull of the individual probabilities.

⁷This is why the pessimistic opinion is a refinement of the cautious opinion, just as the optimistic opinion is a refinement of the bold opinion.

The following theorem proposes a formal characterization of a GMEU, which is a generalization of a GHEU. If society considers only pessimistic and optimistic opinions, then Unanimity implies that society admits for its preferences a consistently utilitarian GMEU representation.

Theorem 4. *Suppose $\Theta = \{\text{pessi}, \text{opti}\}$. Unanimity holds if and only if social preferences $\succeq$ are represented by a consistently utilitarian GMEU with $\mathbb{P} \subseteq \mathbb{P}_\pi$.*

Theorem 5 states that if, additionally, Independence is satisfied, then society must have a consistently utilitarian α -MEU representation.

Theorem 5. *Suppose $\Theta = \{\text{pessi}, \text{opti}\}$. Unanimity and Independence are satisfied if and only if social preferences $\succeq$ are represented by a consistently utilitarian α -MEU with $\mathbb{P} \subseteq \mathbb{P}_\pi$.*

Theorem 5 is a general result, which includes many important results as special cases. If $\mathbb{P}$ is a singleton, all being a convex combination of individual beliefs with $\alpha = 0$, then it coincides with the theorem in Gilboa, Samet and Schmeidler (2004). Similarly, if $\mathbb{P}$ is a set consisting of all convex combinations of individual beliefs with $\alpha = 0$, then it coincides with the result of Alon and Gayer (2016).

4 SOCIAL OPINIONS AND CONTAMINATION

In the previous sections, we only considered social opinions that were based on lotteries or approved acts. However, in some situations, individual beliefs or a weighted average of individual beliefs define undoubtedly a reasonable social opinion. For example, society may be particularly interested in the opinion of a renowned expert. In this case, society may wish to include this expert opinion in the set of possible social opinions.

Definition 13. A social opinion is *Bayesian*, i.e. $\theta = B$, if $\succeq_B$ admits a SEU representation and satisfies HPC.

To be a social opinion, Bayesian preferences $\succeq_B$ must first match with social preferences $\succeq$ regarding lotteries. Moreover, the Bayesian opinion is

also associated to a probability measure p_B on Σ , the algebra of possible events. Since $\succeq_B$ satisfies HPC, we can apply the theorem of [Gilboa, Samet and Schmeidler \(2004\)](#), which immediately implies that Bayesian beliefs p_B are defined as a convex combination of individual beliefs.

Proposition 1 ([Gilboa, Samet and Schmeidler \(2004\)](#)). *If a social opinion is Bayesian, then p_B on Σ is a convex combination of individual beliefs $\{\pi_i\}_{i=1}^n$.*

Therefore, a Bayesian social opinion is also a probabilistic social opinion, i.e. $p_B \in \mathbb{P}_\pi$. However, not all probabilistic social opinions are necessarily Bayesian. We now aim to consider social opinions more holistically rather than separately. According to the principles mentioned above, two sharp opinions, such as cautious and bold opinions, can together form a moderate social opinion. As Theorem 3 demonstrates, a moderate opinion can be represented by a consistent utilitarian HEU.

Definition 14. A social opinion is *moderate*, $\theta = m$, if $\succeq_m$ admits a consistently utilitarian HEU representation.

Let's introduce now the notion of *contamination* when applied to a MEU.

Definition 15. Given $(\mathcal{A}, u, \pi)$, a function $V : \mathcal{F} \rightarrow \mathbb{R}$ is a *contamination* maxmin expected utility (cMEU) if there exist a probability measure p on Σ , a nonempty compact and convex set $\mathbb{P}$ of probabilities on Σ and a utility function u on X such that:

$$(5) \quad V(f) = \epsilon \int u(f) dp + \gamma \max_{q \in \mathbb{P}} \int u(f) dq + (1 - \epsilon - \gamma) \min_{q \in \mathbb{P}} \int u(f) dq,$$

where $\epsilon, \gamma \in [0, 1]$ and $\epsilon + \gamma \leq 1$.

Theorem 6 characterizes cMEU social preferences by satisfying Unanimity and Independence applied to both Bayesian and moderate social opinions.

Theorem 6. *Suppose $\Theta = \{B, m\}$. Unanimity and Independence hold if and only if social preferences $\succeq$ are represented by a consistently utilitarian cMEU with $\mathbb{P} = \mathbb{P}_\pi$ and $p = p_B$.*

Similarly, we could consider another aggregate opinion, called the *neutral* social opinion, which would aggregate the pessimistic and optimistic opinions in the same way that the Bayesian opinion aggregates the cautious and bold opinions. Then, by using Unanimity and Independence with respect to Bayesian and neutral opinions, we could characterize a *consistently moderate expected utility* using a smaller set of social beliefs, that is, $\mathbb{P} \subset \mathbb{P}_\pi$.

It should be noted that our analysis so far focuses solely on a binary opposition of social opinions. What would happen if we expanded the set of social opinions to include all the opinions we discussed earlier, not just in pairs of antagonistic opinions? As observed by [Crès, Gilboa and Vieille \(2011\)](#) and [Qu \(2017\)](#), in the case of multiple opinions, a stronger axiom than Unanimity and Independence would be needed to characterize the representation.

In Theorem 6, we implicitly assume that opinions are aggregated sequentially. We first aggregate cautious and bold opinions into a moderate opinion. Next, we aggregate moderate and Bayesian opinions into cMEU social preferences. However, the order of opinion aggregation does not affect the representation of social preferences. At each step of this sequential aggregation, the binary opposition of the opinions leads to the formation of a new opinion that can be seen as a kind of linear average of the two antagonistic opinions. Therefore, as long as the aggregated opinions stem from a binary opposition, the additive form can be maintained.

5 CONCLUSION

Analyzing the conditions for collective decision-making presents a formidable challenge. Economists have developed numerous models aimed at assisting social planners in making more efficient choices by thoroughly examining the practical problems they encounter. Decision-makers, however, often face a dual complexity: the intricacy of the issues they must tackle, such as analyzing global warming, and the complexity of the economic models required to effectively represent their preferences and available options. This

dual complexity engenders significant structured ambiguity. The primary consequence of this structured ambiguity is that, despite the existence of various models, none can be entirely free from the risk of misspecification.

In light of the risk of model misspecification, the foundational principles upon which traditional social choice theory has long depended—namely, methodological individualism and the Pareto principle—are significantly challenged. Methodological individualism posits that social decision-making should rely solely on the preferences of each individual in society. However, given the risk of model misspecification, it becomes apparent that individual models may harbor errors and potential inaccuracies. Consequently, relying solely on individual preferences to shape social preferences becomes problematic and, indeed, unsuitable.

The unanimity of potentially erroneous individual preferences does not constitute a convincing criterion. Blindly adhering to such unanimity indeed lacks a rational foundation. Therefore, in this article, we contend that strict adherence to the individualistic principle and the Pareto principle is unnecessary once the risk of model misspecification is acknowledged. Instead, we propose an aggregation process based on a principle of *partial* individualism, which is less restrictive than standard methodological individualism.

Our findings suggest that a model of social belief formation centered around approved individual beliefs offers a promising approach to developing and addressing how society can make decisions in the presence of misspecified individual models.

A APPENDIX — PRELIMINARIES AND PROOF OF LEMMA 1

Let $B_0(\Sigma)$ is the vector space generated by the indicator functions of the elements of Σ , endowed with the supnorm. We denote by $ba(\Sigma)$ the set of all bounded, finitely additive set functions on Σ , and by $\Delta(\Sigma)$ the set of all probabilities on Σ . We know that $ba(\Sigma)$, endowed with the total variation norm, is isometrically isomorphic to the norm dual of $B_0(\Sigma)$. Therefore, the weak topology, w^* , of $ba(\Sigma)$ coincides with the eventwise convergence topology. Given a nonsingleton interval K in the real line, $B_0(\Sigma, K)$ is the set of the functions in $B_0(\Sigma)$ taking values in K .

We recall that a binary relation $\succeq$ on $B_0(\Sigma, K)$ is:

- *preordered* if it is reflexive and transitive;
- *continuous* if $\varphi_n \succeq \phi_n$, for all $n \in \mathbb{N}$, $\varphi_n \rightarrow \varphi$ and $\phi_n \rightarrow \phi$ imply $\varphi \succeq \phi$;
- *Archimedean* if the sets $\{\lambda \in [0, 1] : \lambda\varphi + (1 - \lambda)\phi \geq \eta\}$ and $\{\lambda \in [0, 1] : \eta \succeq \lambda\varphi + (1 - \lambda)\phi\}$ are closed in $[0, 1]$, for all $\varphi, \phi, \eta \in B_0(\Sigma, K)$;
- *affine* if, for all $\varphi, \phi, \eta \in B_0(\Sigma, K)$ and all $\alpha \in (0, 1)$, $\varphi \succeq \phi$ iff $\alpha\varphi + (1 - \alpha)\eta \succeq \alpha\phi + (1 - \alpha)\eta$;
- *monotonic* if $\varphi \geq \phi$ implies $\varphi \succeq \phi$;
- *nontrivial* if there exists $\varphi, \phi, \eta \in B_0(\Sigma, K)$ such that $\varphi \succeq \phi$ but not $\phi \succeq \varphi$.

Lemma A1. *A binary relation $\succeq$ is a nontrivial, continuous, affine, and monotonic preorder on $B_0(\Sigma, K)$ iff there exists a nonempty subset $\mathbb{P}$ of $\Delta(\Sigma)$ such that:*

$$(6) \quad \varphi \succeq \phi \iff \int \varphi dp \geq \int \phi dp \quad \text{for all } p \in \mathbb{P}.$$

Moreover, $\overline{co}^{w^*}(\mathbb{P})$ is the unique weak-closed and convex subset of $\Delta(\Sigma)$ representing $\succeq$ in the sense of above expression.

Given a functional $I : B_0(\Sigma) \rightarrow \mathbb{R}$, I is said to be:

- *monotonic* if $I(\varphi) \geq I(\phi)$, for all $\varphi, \phi \in B_0(\Sigma)$ such that $\varphi \geq \phi$;
- *constant additive* if $I(\varphi + a) = I(\varphi) + a$, for all $\varphi \in B_0(\Sigma)$ and $a \in \mathbb{R}$;
- *positively homogeneous* if $I(a\varphi) = aI(\varphi)$, for all $\varphi \in B_0(\Sigma)$ and $a \geq 0$;
- *constant linear* if it is constant additive and positively homogeneous.

A probability measure π is convex-ranged if, for every $0 < r < 1$ and every $A \in \mathcal{A}$, there is a subset $B \subset A$ with $B \in \mathcal{A}$ such that $\pi(B) = r\pi(A)$. Then a countably additive non-atomic measure is convex-ranged. Define now a function $\mu : \Sigma \rightarrow [0, 1]$ such that, for any $E \in \Sigma$: $\mu(E) = \sup\{\pi(A) : A \subset E \text{ and } E \in \mathcal{A}\}$. Since π is countably additive, it is straightforward to show that the supremum can be reached. Call $A \in \mathcal{A}$ the *core* of E if $A \subseteq E$ and $\pi(A) = \mu(E)$ with A unique for a set of zero measure.

Let $A \in \mathcal{A}$, $M = \{1, \dots, m\}$ and $\{B_i\}_{i \in M}$ be a finite partition of A . Let $\mathcal{M}$ be the set of all nonempty subsets of M and define, for $J \in \mathcal{M}$, $\mathcal{M}(J)$ as $\{K \in \mathcal{M} : K \subset J\}$. For $B^J = \cup_{j \in J} B_j$, let C^J be the core of B^J . Note that $C^M = A$. The *unanimous split* $\{\hat{E}^J\}_{J \in \mathcal{M}} \subset \mathcal{A}$ of $\{B_i\}_{i \in M}$ is inductively defined as follows: (1) for all $i \in M$, $\hat{E}^{\{i\}} = C^{\{i\}}$, and (2) for all J such that $|J| > 1$:

$$\widehat{E}^J := C^J \setminus \left(\bigcup_{K \in \mathcal{M}(J), K \neq J} \widehat{E}^K \right).$$

Note that $\{\hat{E}^J\}_{J \in \mathcal{M}}$ is a unanimous partition of A such that $\bigcup_{K \in \mathcal{M}(J)} \widehat{E}^K \subset B^J$, for all $J \in \mathcal{M}$, and $\mu(A^J) = \pi(C^J) = \sum_{K \in \mathcal{M}(J)} \mu(\widehat{E}^K)$. Moreover, for every simple act $f \in \mathcal{F}$ with range $\{x_1, \dots, x_m\}$, let $\{\hat{E}^J(f)\}$ be the *ideal split* of $\{f^{-1}(x_i)\}$.

Lemma A2. *Let $f \in \mathcal{F}$ with range $\{x_1, \dots, x_m\}$. Then, $(f_*, f^*) \in \mathcal{F}^2$ such that $(f_*(s), f^*(s)) = \left(\arg_{x_i} \min_{i \in J} u(x_i), \arg_{x_i} \max_{i \in J} u(x_i) \right)$, for $s \in \widehat{E}^J(f)$, is an envelope of f .*

Lemmas A1-2 are standard. Proofs are then omitted.

Lemma 1. *The collection $\mathcal{A}$ of approved events is a λ -system.*

Proof of Lemma 1. The two first conditions of the definition of a λ -system are obviously satisfied. We only show the condition (iii) holds. Suppose a countable sequence of disjoint events $A_n \in \mathcal{A}$. Since $\mathcal{A}$ is a subset of the σ -algebra Σ , it is clear that the union of the events $A = \bigcup_n A_n$ belongs to Σ . By σ -additivity of every π_i , we have $\pi_i(A) = \sum_n \pi_i(A_n)$. For every A_n and i, j , $\pi_i(A_n) = \pi_j(A_n)$. Hence, if $\pi_i(A) = \pi_j(A)$, for every i, j , it implies that $A \in \mathcal{A}$. $\square$

B APPENDIX — PROOF OF THEOREM 1

Theorem 1. *HPC holds if and only if social preferences $\succsim$ are represented by a consistently utilitarian RSEU.*

The proof of the necessity part is straightforward. We only demonstrate the sufficiency one.

Proof of Theorem 1. Assume HPC holds. We first show that, for all $A \in \mathcal{A}$, if $\pi_i(A) = p \in [0, 1]$, for all i , then $\pi(A) = p$, where $\pi(\cdot)$ represents social beliefs.

Let $p_k = \frac{1}{2^k}$, where $k \in \mathbb{N}$. Take $A \in \mathcal{A}$ such that $\pi_i(A) = p_k$, for all i . We prove now, by induction, that $\pi(A) = p_k$. If $k = 1$, then $\pi_i(A) = \frac{1}{2}$ and $A \in \mathcal{A}$. We claim that $\pi(A) = \frac{1}{2}$. Suppose it is wrong, and wlog (*without loss of generality*) assume that $\pi(A) > \frac{1}{2}$. Therefore, there exist $x, y \in X$ such that $xAy \succ xA^c y$. However, for all i , $xAy \sim_i xA^c y$, which, by HPC, implies: $xAy \sim xA^c y$, i.e. a contradiction. A similar argument works for the case where $\pi(A) < \frac{1}{2}$. Hence, $\pi_i(A) = \frac{1}{2}$, for all i , implies that $\pi(A) = \frac{1}{2}$. Now, suppose that $\pi_i(A) = \frac{1}{2^k}$, for all i , implies that $\pi(A) = \frac{1}{2^k}$. Assume that $\pi_i(A) = \frac{1}{2^{k+1}}$, for all i . We then want to show that $\pi(A) = \frac{1}{2^{k+1}}$. Suppose it is wrong and wlog assume that $\pi(A) > \frac{1}{2^{k+1}}$. By Lyapunov Convexity Theorem (See (Artstein, 1990) for a formal proof), there exists a subset $B \subset A^c$ such that $\pi_i(B) = \frac{1}{2^{k+1}}$, for all i . So, $B \in \mathcal{A}$ and $A \cap B = \emptyset$ imply that $A \cup B \in \mathcal{A}$. Since $\pi_i(A \cup B) = \frac{1}{2^k}$, for all i , by assumption, we have $\pi(A \cup B) = \frac{1}{2^k}$, which means $\pi(B) < \frac{1}{2^{k+1}}$. Similarly, there exists a subset $C \subset (A \cup B)^c$ s.t. $\pi_i(C) = \frac{1}{2^{k+1}}$, for all i . Hence, $\pi_i(A \cup C) = \frac{1}{2^k} = \pi_i(B \cup C)$, for all i , which implies that $\pi(A \cup C) = \frac{1}{2^k} =$

$\pi(B \cup C)$. However, the first equality means that $\pi(C) < \frac{1}{2^{k+1}}$, while the second equality means that $\pi(C) > \frac{1}{2^{k+1}}$, that is a contradiction. The same argument works for the case where $\pi(A) < \frac{1}{2^{k+1}}$.

Now, first, take an arbitrary rational number $p \in (0, 1)$. Then p admits a finite dyadic expansion:

$$p = \sum_{k=1}^m \frac{x_k}{2^k},$$

where $x_k \in \{0, 1\}$. Take $A \in \mathcal{A}$ s.t. $\pi(A) = p$, for all i . Therefore, there exists a partition $\{A_1, \dots, A_m\}$ of A s.t. $\pi_i(A_k) = \frac{x_k}{2^k}$, for all i and $k = 1, \dots, m$. Thanks to the above analysis, we have $\pi(A_k) = \frac{x_k}{2^k}$, for all $k = 1, \dots, m$. It is immediate to see that $\pi(A) = p$. Second, take an arbitrary irrational number $p \in (0, 1)$ and A s.t. $\pi_i(A) = p$, for all i . Suppose $\pi(A) \neq p$ and assume $\pi(A) > p$. There exists a rational number q s.t. $\pi(A) > q > p$. We can find an event B s.t. $A \subset B$ and $\pi_i(B) = q$, for all i . This requires that $\pi(B) = q < \pi(A)$, which contradicts the fact that $A \subset B$ implies $\pi(A) \leq \pi(B)$. A similar argument works for the case where $\pi(A) < p$. Hence, finally, $\pi(A) = p$.

We show now that the social utility u is a convex combination of individual utilities. Note that, for any nonnegative numbers $p_1, \dots, p_m$ s.t. $\sum_{k=1}^m p_k = 1$, there exists a partition $\{A_k\}_{k=1}^m$ of S s.t. $\pi(A_k) = \pi_i(A_k) = p_k$, for all i and k . Therefore, for any vNM lottery L defined over X , we can construct an act $f \in \mathcal{L}$ s.t. the lottery L corresponds to the distribution on X generated by f . Conversely, any finitely valued act $f \in \mathcal{L}$ defines a distribution over X , which is a vNM lottery. In restricting preferences over $\mathcal{L}$, we can apply Harsanyi Theorem to conclude that u is a convex combination of $\{u_i\}_{i=1}^n$. $\square$

C APPENDIX — PROOF OF THEOREM 2

Several notions and intermediate results are necessary.

A set function $\nu : \Sigma \rightarrow [0, 1]$ is a *capacity* if $\nu(\emptyset) = 0$, $\nu(S) = 1$ and $A \subseteq B$ implies $\nu(A) \leq \nu(B)$. Given π on $\mathcal{A}$, we define set functions $\mu_*, \mu^* : \Sigma \rightarrow [0, 1]$

by: for $A \in \Sigma$,

$$(7) \quad \mu_*(A) = \sup_{\substack{B \subset A \\ B \in \mathcal{A}}} \{\pi(B)\} \quad \text{and} \quad \mu^*(A) = \inf_{\substack{A \subset B \\ B \in \mathcal{A}}} \{\pi(B)\}.$$

Lemma C1. μ_* and μ^* are capacities on Σ .

We omit the proof since it is straightforward.

Recall now the notion of Choquet integration. For any capacity ν and integrand $a : S \rightarrow \mathbb{R}$, the *Choquet integral* is defined by:

$$\int a d\nu = \int_0^\infty \nu(\{s : a(s) \geq t\}) dt + \int_{-\infty}^0 [\nu(\{s : a(s) \geq t\}) - 1] dt.$$

Therefore, a function $V : \mathcal{F} \rightarrow \mathbb{R}$ is said to be a *Choquet expected utility* (CEU) function if there exist a function u on X and a capacity μ on Σ s.t., for $f \in \mathcal{F}$:

$$V(f) = \int u(f) d\mu.$$

Lemma C2. The cautious social opinion $\succsim_{\text{caut}}$ is represented by a function $V_{\text{caut}} : \mathcal{F} \rightarrow \mathbb{R}$, where, for all $f \in \mathcal{F}$: $V_{\text{caut}}(f) = \min_{p \in \mathbb{P}_\pi} \int u(f) dp$. Similarly, the bold social opinion $\succsim_{\text{bold}}$ is represented by $V_{\text{bold}} : \mathcal{F} \rightarrow \mathbb{R}$, where, for all $f \in \mathcal{F}$: $V_{\text{bold}}(f) = \max_{p \in \mathbb{P}_\pi} \int u(f) dp$.

Proof of Lemma C2. We will prove the result for V_{caut} . The proof for V_{bold} is analogous and, therefore, omitted.

First, show that for $f \in \mathcal{F}$:

$$V_{\text{caut}}(f) = \sup \left\{ \int u(g) d\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L} \right\}.$$

By monotonicity, for all $g \in \mathcal{L}$, if $u(f) \geq u(g)$, then $V_{\text{caut}}(f) \geq \int u(g) d\pi$. Therefore, it is clear that $V_{\text{caut}}(f) \geq \sup \left\{ \int u(g) d\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L} \right\}$. Now, suppose that $V_{\text{caut}}(f) > \sup \left\{ \int u(g) d\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L} \right\}$. We want now to derive a contradiction. Since f is a simple act, there exist x, y in $\{x_1, \dots, x_m\}$, which is the outcome set of the act f , s.t. $u(x) \geq u(z) \geq u(y)$,

for all $z \in \{x_1, \dots, x_m\}$. Therefore, $x \succeq f \succeq y$. Since $u(X)$ is convex, we know there exists $x_f \in X$ s.t. $x_f \sim f$, which implies $u(x) > \sup\{\int u(g)d\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L}\}$. Again, by convexity of $u(X)$, there is a $y \in X$ such that: $u(x) > u(y) > \sup\{\int u(g)d\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L}\}$. As a result, $f \succeq_{\text{caut}} y$ while there is no $g \in \mathcal{L}$ s.t. $u(f) \geq u(g)$ and $g \succ y$, which contradicts the definition of the cautious social opinion. Therefore, for $f \in \mathcal{F}$, $V_{\text{caut}}(f) = \sup\{\int u(g)d\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L}\}$.

Second, we want to show that, for $f \in \mathcal{F}$,

$$\int u(f)d\mu = \sup\{\int u(g)d\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L}\}.$$

Notice that, according to [Schmeidler \(1989\)](#), CEU satisfies monotonicity, which means, if $u(f) \geq u(g)$, that $\int u(f)d\mu \geq \int u(g)d\mu$. When $g \in \mathcal{L}$, $\int u(g)d\mu = \int u(g)d\pi$. Therefore, we have: $\int u(f)d\mu \geq \sup\{\int u(g)d\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L}\}$. Take an act $f \in \mathcal{F}$. Wlog, we can write $f = x_1 A_1 x_2 A_2 \dots x_m A_m$, where $u(x_1) > u(x_2) > \dots > u(x_m)$. So,

$$\int u(f)d\mu = \sum_{k=1}^{m-1} [u(x_k) - u(x_{k+1})] \mu(\cup_{j=1}^k A_j) + u(x_m).$$

Let $\widehat{E}^J(f)$ be the ideal split of $\{f^{-1}(x_i)\}$. Consider $g \in \mathcal{L}$ defined by, for all $s \in S$, $g(s) = \arg \min_{i \in J} u(x_i)$ if $s \in \widehat{E}^J(f)$. We want to show that:

$$\int u(g)d\pi = \sup\left\{\int u(g)d\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L}\right\}.$$

For $1 \leq k \leq m$, we write $\widehat{E}_{\leq k}^J(f) := \cup_J \{\widehat{E}^J(f) : J \subseteq \{1, 2, \dots, k\} \text{ and } k \in J\}$. Since $x_1 > x_2 > \dots > x_m$, we can rewrite g in the following way: for all $s \in S$, $g(s) = x_k$ if $s \in \widehat{E}_{\leq k}^J(f)$. We know that, for every $1 \leq k \leq m$, $\sum_{j=1}^k \pi(\widehat{E}_{\leq j}^J(f)) =$

$\pi(C^{\{1, \dots, k\}})$, which implies $\pi(\widehat{E}_{\leq k}^J(f)) = \pi(C^{\{1, \dots, k\}}) - \pi(C^{\{1, \dots, k-1\}})$. Therefore:

$$\begin{aligned} V_{\text{caut}}(g) &= \sum_{k=1}^m u(x_k) \pi(\widehat{E}_{\leq k}^J(f)) \\ &= \sum_{k=1}^m u(x_k) [\pi(C^{\{1, \dots, k\}}) - \pi(C^{\{1, \dots, k-1\}})] \\ &= \sum_{k=1}^m u(x_k) [\mu_*(\cup_{j=1}^k A_j) - \mu_*(\cup_{j=1}^{k-1} A_j)] \\ &= \int u(f) d\mu. \end{aligned}$$

Hence, there exists $p_* \in \mathbb{P}_\pi$ s.t. $p_*(A_k) = \mu_*(\cup_{j=1}^k A_j) - \mu_*(\cup_{j=1}^{k-1} A_j)$. That is, $\int u(f) d\mu = \int u(f) dp_* \geq \min_{p \in \mathbb{P}_\pi} \int u(f) dp$. However, $u(f) \geq u(g)$ implies that, for all $p \in \mathbb{P}_\pi$, $\int u(f) dp \geq \int u(g) dp = \int u(g) d\pi$. So, $\min_{p \in \mathbb{P}_\pi} \int u(f) dp \geq \int u(g) d\pi = \int u(f) d\mu$. Hence, we have $\int u(f) d\mu = \min_{p \in \mathbb{P}_\pi} \int u(f) dp$. $\square$

Recall now that $\bar{u}_{\mathbb{P}}^f \equiv \max_{p \in \mathbb{P}} \int u(f) dp$ and $\underline{u}_{\mathbb{P}}^f \equiv \min_{p \in \mathbb{P}} \int u(f) dp$.

Theorem 2. Suppose $\Theta = \{\text{caut}, \text{bold}\}$. Unanimity holds if and only if social preferences $\succsim$ are represented by a consistently utilitarian GHEU.

Proof of Theorem 2. The proof of the necessity part is straightforward. We only prove the sufficiency one. From previous analysis, we know that, for $f, g \in \mathcal{F}$, $f \succsim_{\text{caut}} g$ iff $\underline{u}_{\mathbb{P}_\pi}^f \geq \underline{u}_{\mathbb{P}_\pi}^g$ and $f \succsim_{\text{bold}} g$ iff $\bar{u}_{\mathbb{P}_\pi}^f \geq \bar{u}_{\mathbb{P}_\pi}^g$. Thus, Unanimity implies $f \succsim g$, whenever $\underline{u}_{\mathbb{P}_\pi}^f \geq \underline{u}_{\mathbb{P}_\pi}^g$ and $\bar{u}_{\mathbb{P}_\pi}^f \geq \bar{u}_{\mathbb{P}_\pi}^g$. Therefore, there exists a monotonic function $W : u(X) \times u(X) \rightarrow \mathbb{R}$ s.t. $W(\underline{u}_{\mathbb{P}_\pi}^f, \bar{u}_{\mathbb{P}_\pi}^f) = W(\underline{u}_{\mathbb{P}_\pi}^g, \bar{u}_{\mathbb{P}_\pi}^g)$, whenever $\underline{u}_{\mathbb{P}_\pi}^f = \underline{u}_{\mathbb{P}_\pi}^g$ and $\bar{u}_{\mathbb{P}_\pi}^f = \bar{u}_{\mathbb{P}_\pi}^g$. Thus, for any act f , the associated pair $(\underline{u}_{\mathbb{P}_\pi}^f, \bar{u}_{\mathbb{P}_\pi}^f)$ characterizes the indifference class with respect to f . Hence, W is a representation of $\succsim$ on $\mathcal{F}$. $\square$

D APPENDIX — PROOF OF THEOREM 3

Theorem 3. *Suppose $\Theta = \{\text{caut}, \text{bold}\}$. Unanimity and Independence hold if and only if social preferences $\succeq$ are represented by a consistently utilitarian HEU.*

The necessity part is straightforward and, therefore, omitted. Then, we just show the sufficiency part. The proof consists of demonstrating, step by step, three intermediate lemmas, i.e. Lemma D1-3. Observe first that $\{u(f) : f \in \mathcal{F}\} = \{\phi \in B_0(\Sigma) : \phi = u(f), \text{ for some } f \in \mathcal{F}\} = B_0(\Sigma, u(X))$. Wlog, assume that $[-1, 1] \subset u(X)$. Define I on $B_0(\Sigma, u(X))$ as follows: for all $f \in \mathcal{F}$, $I(u(f)) = V(f)$. Note that $f \succeq g$ iff $I(u(f)) \geq I(u(g))$, for all $f, g \in \mathcal{F}$. Moreover, $I(\mathbf{1}) = 1$.

Lemma D1. *I is positively homogeneous.*

Proof of Lemma D1. For $\varphi \in B_0(\Sigma, u(X))$ and $a \geq 0$, show that $I(a \cdot \varphi) = aI(\varphi)$. Let $f \in \mathcal{F}$ be an act s.t. $I(\varphi) = V(f)$. Let $x_0 \in X$ be defined by $u(x_0) = 0$. By continuity and monotonicity, there exists $x \in X$ s.t. $u(x) = V(f)$. Consider $a \in (0, 1)$. Thanks to the convexity of π , there exists a subset $A \in \mathcal{A}$ s.t. $\pi(A) = a$. Let $g \in \mathcal{F}$ be defined as follows: for all $s \in S$, $g(s) \sim f(s)Ax_0$. Since $u(g(s)) = a \cdot u(f(s))$, for all s , we have: $V(g) = I(a\varphi)$. Furthermore, note that f and g admit the same ideal splitting. Therefore, $V^*(g) = V^*(f[A]x_0)$ and $V_*(g) = V_*(f[A]x_0)$, that is:

$$\int u(g) d\mu^* = a \cdot \int u(f) d\mu^* \quad \text{and} \quad \int u(g) d\mu_* = a \cdot \int u(f) d\mu_*.$$

Hence, if $f \sim^* x^*$ and $f \sim_* x_*$, then $g \sim^* x^*Ax_0$ and $g \sim_* x_*Ax_0$. By Independence, we have $g \sim xAx_0$, which means $V(g) = a \cdot u(x) = a \cdot V(f)$. Hence, $I(a\varphi) = aI(\varphi)$, for $a \in (0, 1)$. If $a = 0$ or $a = 1$, the result holds trivially. If $a > 1$, then $\frac{1}{a}I(a \cdot \varphi) = I(\varphi)$ according to the above argument. This ends the proof. $\square$

It is now sufficient to extend I by homogeneity to all $B_0(\Sigma)$. Note that I is monotone and positively homogeneous on $B_0(\Sigma)$.

Lemma D2. *I is constant additive.*

Proof of Lemma D2. Let $\varphi \in B_0(\Sigma)$ and $a \in \mathbb{R}$. We want to show $I(\varphi + a \cdot \mathbb{1}) = I(\varphi) + a$. Let $f \in \mathcal{F}$ be s.t. $u(f) = 2\varphi$ and $x \in X$ be s.t. $u(x) = 2a$. Also, by continuity and monotonicity, there is $y \in X$ s.t. $f \sim y$. By convexity of π , take $A \in \mathcal{A}$ s.t. $\pi(A) = \frac{1}{2}$. Define act $g \in \mathcal{F}$ by for all s , $g(s) \sim f(s)Ax$. So, $u(g(s)) = \frac{u(f(s)) + u(x)}{2}$ for all s , which implies $u(g) = \varphi + a \cdot \mathbb{1}$. Since f and g have identical ideal splitting, we must have $g \sim^* f[A]x$ and $g \sim_* f[A]x$. Therefore,

$$\int u(g) d\mu^* = \frac{1}{2} \left[\int u(f) d\mu^* + u(x) \right] \quad \text{and} \quad \int u(g) d\mu_* = \frac{1}{2} \left[\int u(f) d\mu_* + u(x) \right].$$

Let $y^* \sim^* f$ and $y_* \sim_* f$. Then, $g \sim^* y^*Ax$ and $g \sim_* y_*Ax$. According to Independence, we have $g \sim yAx$. Therefore, $I(\varphi + a \cdot \mathbb{1}) = \frac{1}{2}(u(y) + u(x)) = \varphi + a$. $\square$

Let $B_0(\mathcal{A})$ denote the set of all real-valued $\mathcal{A}$ -measurable finite valued functions. For $\varphi \in B_0(\Sigma)$, let

$$\varphi^* = \arg \inf_{\substack{\phi \in B_0(\mathcal{A}) \\ \phi \geq \varphi}} I(\phi) \quad \text{and} \quad \varphi_* = \arg \sup_{\substack{\phi \in B_0(\mathcal{A}) \\ \varphi \geq \phi}} I(\phi)$$

Note that, for $f \in \mathcal{F}$, $V_g(f) = I(u(f)^*)$ and $V_c(f) = I(u(f)_*)$.

Lemma D3. *Let $I : B_0(\Sigma) \rightarrow \mathbb{R}$ be a monotonic constant linear functional. Then, there exists a unique $\alpha \in [0, 1]$ such that, for all $\varphi \in B_0(\Sigma)$, $I(\varphi) = \alpha I(\varphi^*) + (1 - \alpha)I(\varphi_*)$.*

Proof of Lemma D3. By Theorem 2, we know that $I(\varphi) = W(I(\varphi_*), I(\varphi^*))$. Since I is homogeneous and constant additive, we have, for $\alpha \in [0, 1]$ and $a \in \mathbb{R}$:

$$\begin{aligned} W(\alpha\varphi_*, \alpha\varphi^*) &= \alpha W(\varphi_*, \varphi^*), \\ W(\varphi_* + a \cdot \mathbb{1}, \varphi^* + a \cdot \mathbb{1}) &= W(\varphi_*, \varphi^*) + a. \end{aligned}$$

Note that $I(\varphi - I(\varphi_*)) = W(I(\varphi_*) - I(\varphi_*), I(\varphi^*) - I(\varphi_*)) = W(0, I(\varphi^*) - I(\varphi_*))$.

Therefore:

$$\begin{aligned} W(I(\varphi_*), I(\varphi^*)) &= W(0, I(\varphi^*) - I(\varphi_*)) + I(\varphi_*) \\ &= W(0, 1)(I(\varphi^*) - I(\varphi_*)) + I(\varphi_*). \end{aligned}$$

Let $\alpha = W(0, 1)$. We have: $I(\varphi) = \alpha I(\varphi^*) + (1 - \alpha)I(\varphi_*)$. Let also φ be s.t. $I(\varphi_*) = 0$. Then, monotonicity implies that $I(\varphi) = \alpha > 0$. $\square$

This ends the proof of Theorem 3.

E APPENDIX — PROOF OF THEOREM 4 AND 5

The proofs of Theorem 4 and 5 proceed in six steps, corresponding to six lemmas, i.e. Lemma E1-6. Suppose $\Theta = \{\text{opti}, \text{pessi}\}$. Lemma E1-3, which do not assume Independence, are sufficient for the proof of Theorem 4. Lemma 4-6, which assume Independence, are used to derive Theorem 5.

The necessity of both theorems are standard, we hence omit it.

Lemma E1. *There exists a unique non-empty convex and compact set $\mathbb{P} \subseteq \mathbb{P}_\pi$ of probabilities on Σ such that, for all $f, g \in \mathcal{F}$:*

$$f \succeq g \iff \int u(f)dp \geq \int u(g)dp \quad \text{for all } p \in \mathbb{P}.$$

Proof of Lemma E1. By definition of the binary relation $\succeq$, we know, for all $h \in \mathcal{F}$ and $A \in \mathcal{A}$, that:

$$\begin{aligned} f \succeq g &\iff I(u(f[A]h)) \geq I(u(g[A]h)) \\ &\iff I(\pi(A)u(f) + (1 - \pi(A))u(h)) \geq I(\pi(A)u(g) + (1 - \pi(A))u(h)). \end{aligned}$$

Since π has a convex range on $\mathcal{A}$, for each $\lambda \in [0, 1]$, there exists $A \in \mathcal{A}$ s.t. $\lambda = \pi(A)$. Therefore, for all $\lambda \in (0, 1)$ and $h \in \mathcal{F}$: $f \succeq g \iff I(\lambda u(f) + (1 - \lambda)u(h)) \geq I(\lambda u(g) + (1 - \lambda)u(h))$. Now, we define $\geq$ on $B_0(\Sigma, u(X))$ as follows: for all $\varphi, \psi \in B_0(\Sigma, u(X))$: $\varphi \geq \psi \iff I(\lambda \varphi + (1 - \lambda)\psi) \geq I(\lambda \psi + (1 - \lambda)\varphi), \forall \psi \in B_0(\Sigma, u(X)), \lambda \in (0, 1]$. Hence, it is straightforward that $f \succeq$

$g \Leftrightarrow u(f) \geq u(g)$. Therefore, $\geq$ is obviously a non-trivial, monotonic and conic preorder on $B_0(\Sigma, u(X))$. According to [Bewley \(2002\)](#) or [Ghirardato, Maccheroni and Marinacci \(2004\)](#), we know that there exists a unique non-empty convex and compact set $\mathbb{P}$ of probabilities on Σ s.t., for all $\varphi, \phi \in B_0(\Sigma, u(X))$: $\varphi \geq \phi \iff \int \varphi dp \geq \int \phi dp$, for all $p \in \mathbb{P}$. We are left to show that $\mathbb{P} \subseteq \mathbb{P}_\pi$. Suppose it is wrong, i.e. there exists $p \in \mathbb{P}$ s.t. $p \notin \mathbb{P}_\pi$. Since $\mathbb{P}_\pi$ contains all extensions of π , p is not an extension of π . So there exists $A \in \mathcal{A}$ s.t. $p(A) \neq \pi(A)$. Wlog, assume $p(A) \geq \pi(A)$. Then, for $u(x) > u(y)$, we have: $u(x)p(A) + u(y)(1 - p(A)) > u(x)\pi(A) + u(y)(1 - \pi(A))$. By continuity of u , there exists $z \in X$ such that: $u(x)p(A) + u(y)(1 - p(A)) > u(z) > u(x)\pi(A) + u(y)(1 - \pi(A))$. Therefore, $z \succeq_\theta xAy$, for all $\theta \in \{\text{opti}, \text{pessi}\}$ and $z \not\succeq xAy$, which contradicts Unanimity. $\square$

Lemma E2. *For each $\varphi \in B_0(\Sigma, u(X))$, we have:*

$$\min_{p \in \mathbb{P}} \int \varphi dp \leq I(\varphi) \leq \max_{p \in \mathbb{P}} \int \varphi dp.$$

Proof of Lemma E2. The proof is made by negation. First, suppose that there exists $\varphi \in B_0(\Sigma, u(X))$ s.t. $I(\varphi) < \min_{p \in \mathbb{P}} \int \varphi dp$. Let $f \in \mathcal{F}$ and $x \in X$ be s.t. $u(f) = \varphi$ and $u(x) = \phi$. Suppose that $f \sim x$. Then, $I(\varphi) = I(\phi)$. However, for all $\lambda \in (0, 1]$ and $\psi \in B_0(\Sigma, u(X))$, we have: $I(\lambda\varphi + (1 - \lambda)\psi) > I(\lambda\phi + (1 - \lambda)\psi)$, which implies that $f \succ x$. This contradicts the assumption whereby $f \sim x$. A similar argument works when $I(\varphi) > \max_{p \in \mathbb{P}} \int \varphi dp$. $\square$

Lemma E3. *The optimistic social opinion $\succeq_{\text{opti}}$ is represented by V_{opti} s.t., for $f \in \mathcal{F}$, $V_{\text{opti}}(f) = \max_{p \in \mathbb{P}} \int u(f) dp$. Moreover, the pessimistic social opinion $\succeq_{\text{pessi}}$ is represented by V_{pessi} s.t., for $f \in \mathcal{F}$, $V_{\text{pessi}}(f) = \min_{p \in \mathbb{P}} \int u(f) dp$.*

Proof of Lemma E3. Prove the result for V_{opti} . The proof for V_{pessi} is analogous and, therefore, omitted. Note that $\succeq_{\text{opti}}$ admits a restricted SEU representation. So, by monotonicity and continuity, for each act $f \in \mathcal{F}$, there exists $x_f \in X$ s.t. $x_f \sim_{\text{opti}} f$. If $x_f \not\succeq f$, then there exist $g \in \mathcal{F}$ and $A \in \mathcal{A}$ s.t. $f[A]g \succ x_f[A]g$. By definition of $\succeq_{\text{opti}}$, we have $f \succ_{\text{opti}} x_f$. Hence, $x_f \succeq f$. This implies that $u(x_f) \geq \int u(f) dp$, for all $p \in \mathbb{P}$, which

means $u(x_f) \geq \max_{p \in \mathbb{P}} \int u(f) dP$. Now, suppose that, for some $f \in \mathcal{F}$, $u(x_f) > \max_{p \in \mathbb{P}} \int u(f) dP$. Take $x_{\min}$ be s.t. $u(f(s)) \geq u(x_{\min})$, for all s . Then, there exists an event $A \in \mathcal{A}$ such that: $u(x_f) > \max_{p \in \mathbb{P}} \int u(f) dP = u(x_f A x_{\min})$. Let $z \in X$ be s.t. $z \sim x_f A x_{\min}$. We have: $x_f \succ_{\text{opti}} z \succeq f$, which then implies $x_f \succ_{\text{opti}} f$, that is a contradiction. In conclusion, $V_{\text{opti}}(f) = \max_{p \in \mathbb{P}} \int u(f) dP$, represents $\succeq_{\text{opti}}$. $\square$

Theorem 4. Suppose $\Theta = \{\text{pessi}, \text{opti}\}$. Unanimity holds if and only if social preferences $\succeq$ are represented by a consistently utilitarian GMEU with $\mathbb{P} \subseteq \mathbb{P}_\pi$.

Proof of Theorem 4. As seen previously, we know that, for $f, g \in \mathcal{F}$, $f \succeq_{\text{pessi}} g$ iff $\underline{u}_{\mathbb{P}}^f \geq \underline{u}_{\mathbb{P}}^g$ and $f \succeq_{\text{opti}} g$ iff $\bar{u}_{\mathbb{P}}^f \geq \bar{u}_{\mathbb{P}}^g$. Thus, Unanimity implies $f \succeq g$, whenever $\underline{u}_{\mathbb{P}}^f \geq \underline{u}_{\mathbb{P}}^g$ and $\bar{u}_{\mathbb{P}}^f \geq \bar{u}_{\mathbb{P}}^g$. Therefore, there exists a monotonic function $W : u(X) \times u(X) \rightarrow \mathbb{R}$ s.t. $W(\underline{u}_{\mathbb{P}}^f, \bar{u}_{\mathbb{P}}^f) = W(\underline{u}_{\mathbb{P}}^g, \bar{u}_{\mathbb{P}}^g)$, whenever $\underline{u}_{\mathbb{P}}^f = \underline{u}_{\mathbb{P}}^g$ and $\bar{u}_{\mathbb{P}}^f = \bar{u}_{\mathbb{P}}^g$. Thus, for any act f , the associated pair $(\underline{u}_{\mathbb{P}}^f, \bar{u}_{\mathbb{P}}^f)$ characterizes the indifference class with respect to f . Hence, W is a representation of $\succeq$ on $\mathcal{F}$. $\square$

Theorem 5. Suppose $\Theta = \{\text{pessi}, \text{opti}\}$. Unanimity and Independence hold if and only if social preferences $\succeq$ are represented by a consistently utilitarian α -MEU with $\mathbb{P} \subseteq \mathbb{P}_\pi$.

The proof is based on the three following lemmas:

Lemma E4. I is positively homogeneous.

Proof of Lemma E4. For $\varphi \in B_0(\Sigma, u(X))$ and $a \geq 0$, show that $I(a \cdot \varphi) = aI(\varphi)$. Let $f \in \mathcal{F}$ be s.t. $I(\varphi) = V(f)$ and let $x_0 \in X$ be s.t. $u(x_0) = 0$. By continuity and monotonicity, there exists $x \in X$ s.t. $u(x) = V(f)$. Take now $a \in (0, 1)$. Because of the convexity of π , there exists $A \in \mathcal{A}$ s.t. $\pi(A) = a$. Let an act $g \in \mathcal{F}$ be defined by, for all $s \in S$, $g(s) \sim f(s) A x_0$. Since $u(g(s)) = a \cdot u(f(s))$, for all s , we have $V(g) = I(a\varphi)$. Furthermore, note that V_{opti} and V_{pessi} satisfy homogeneity and constant additivity. Therefore, $V_{\text{opti}}(g) = aV_{\text{opti}}(f)$ and $V_{\text{pessi}}(g) = aV_{\text{pessi}}(f)$. Hence, let $x^*, x_* \in X$ be s.t. $f \sim_{\text{opti}} x^*$ and $f \sim_{\text{pessi}} x_*$. Then, $g \sim_{\text{opti}} x^* A x_0$ and $g \sim_{\text{pessi}} x_* A x_0$. By Independence, we have $g \sim x A x_0$,

which leads to $V(g) = a \cdot u(x) = a \cdot V(f)$. Thus, $I(a\varphi) = aI(\varphi)$, for $a \in (0, 1)$. If $a = 0$ or $a = 1$, the result holds trivially. If $a > 1$, then $\frac{1}{a}I(a \cdot \varphi) = I(\varphi)$ according to the above argument. This ends the proof. $\square$

By homogeneity, we now extend I to all $B_0(\Sigma)$. Note that I is monotone and positively homogeneous on $B_0(\Sigma)$.

Lemma E5. *I is constant additive.*

Proof of Lemma E5. Let $\varphi \in B_0(\Sigma)$ and $a \in \mathbb{R}$. We want to show that $I(\varphi + a \cdot \mathbb{1}) = I(\varphi) + a$. Let $f \in \mathcal{F}$ be s.t. $u(f) = 2\varphi$ and $x \in X$ be s.t. $u(x) = 2a$. In addition, by continuity and monotonicity, there is $y \in X$ s.t. $f \sim y$. By the convexity of π , take $A \in \mathcal{A}$ s.t. $\pi(A) = \frac{1}{2}$. Define act $g \in \mathcal{F}$ by, for all s , $g(s) \sim f(s)Ax$. Hence, $u(g(s)) = \frac{u(f(s)) + u(x)}{2}$, for all s , which implies $u(g) = \varphi + a \cdot \mathbb{1}$. Since V_{opti} and V_{pessi} are constant additive, we have $V_{\text{opti}}(g) = \frac{1}{2}V_{\text{opti}}(f) + a$ and $V_{\text{pessi}}(g) = \frac{1}{2}V_{\text{pessi}}(f) + a$. Let $y^*, y_* \in X$ be s.t. $f \sim_{\text{opti}} y^*$ and $f \sim_{\text{pessi}} y_*$. Then, $g \sim_{\text{opti}} y^*Ax$ and $g \sim_{\text{pessi}} y_*Ax$. By Independence, we have $g \sim yAx$. Therefore, $I(\varphi + a \cdot \mathbb{1}) = \frac{1}{2}(u(y) + u(x)) = \varphi + a$. $\square$

Lemma E6. *There exists a unique $\alpha \in [0, 1]$ s.t. $I(\varphi) = \alpha I^*(\varphi) + (1 - \alpha)I_*(\varphi)$.*

Proof of Lemma E6. Since I satisfies homogeneity and constant additivity, according to Lemma E5, there exists a unique $\alpha \in [0, 1]$ s.t. the above expression holds. $\square$

This ends the proof of Theorem 5.

F APPENDIX — PROOF OF THEOREM 6

Theorem 6. *Suppose $\Theta = \{B, m\}$. Unanimity and Independence hold if and only if social preferences $\succeq$ are represented by a consistently utilitarian cMEU with $\mathbb{P} = \mathbb{P}_\pi$ and $p = p_B$.*

Since the necessity part is straightforward, we only prove the sufficiency one.

Proof of Theorem 6. For all acts $f, g \in \mathcal{F}$, $f \succeq_B g$ iff $u_{p_B}^f \geq u_{p_B}^g$ and $f \succeq_m g$ iff $\alpha \bar{u}_{\mathbb{P}_\pi}^f + (1-\alpha) \underline{u}_{\mathbb{P}_\pi}^f \geq \alpha \bar{u}_{\mathbb{P}_\pi}^g + (1-\alpha) \underline{u}_{\mathbb{P}_\pi}^g$. Unanimity implies the existence of $W : u(X) \times u(X) \rightarrow \mathbb{R}$ s.t.: $W(u_{p_B}^f, \alpha \bar{u}_{\mathbb{P}_\pi}^f + (1-\alpha) \underline{u}_{\mathbb{P}_\pi}^f) \geq W(u_{p_B}^g, \alpha \bar{u}_{\mathbb{P}_\pi}^g + (1-\alpha) \underline{u}_{\mathbb{P}_\pi}^g)$, whenever $u_{p_B}^f \geq u_{p_B}^g$ and $\alpha \bar{u}_{\mathbb{P}_\pi}^f + (1-\alpha) \underline{u}_{\mathbb{P}_\pi}^f \geq \alpha \bar{u}_{\mathbb{P}_\pi}^g + (1-\alpha) \underline{u}_{\mathbb{P}_\pi}^g$. Hence, $V(f) = W(u_{p_B}^f, \alpha \bar{u}_{\mathbb{P}_\pi}^f + (1-\alpha) \underline{u}_{\mathbb{P}_\pi}^f)$ represents social preferences $\succeq$. Since $\succeq_B$ and $\succeq_m$ are constantly independent, a similar argument can be used as in Lemmas E4 and E5 and Independence implies that I defined as $I(u(f)) = V(f)$ is homogeneous and constantly additive. Therefore, Lemma D3 yields the existence of a unique $\varepsilon \in [0, 1]$ s.t., for $f \in \mathcal{F}$, $V(f) = \varepsilon u_{p_B}^f + (1-\varepsilon)(\alpha \bar{u}_{\mathbb{P}_\pi}^f + (1-\alpha) \underline{u}_{\mathbb{P}_\pi}^f)$. $\square$

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