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# Weak Agreement and the Properties of Beliefs under Ambiguity

**Antoine Billot**

Université Paris-Panthéon-Assas, LEMMA

**Vassili Vergopoulos**

Université Paris-Panthéon Assas, LEMMA

LEMMA EA 4442 - 4 rue Blaise Desgoffe 75006 Paris

# Weak Agreement and the Properties of Beliefs under Ambiguity*

Antoine Billot[†] and Vassili Vergopoulos[‡]

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**Abstract**

We extend the Agreement Theorem of Aumann [1976] in two key directions. First, we introduce Knightian uncertainty by modeling beliefs as sets of probability measures, allowing for ambiguity in agents' posterior beliefs. Second, we relax the assumption that agents observe perfectly each other's posterior probabilities, replacing it with the assumption that they perceive only certain properties of their posterior probability sets. Our main result establishes that if agents share at least one common prior, they can only have common knowledge that their posterior probabilities satisfy a given property if these properties are mutually compatible. Furthermore, we explore economic implications in the context of trade under asymmetric information, deriving a No Trade result under ambiguity and highlighting conditions under which trade may still occur.

**Keywords:** Agreement, ambiguity, multiple priors, properties, trade.

**JEL classification:** C70, D81, D83.

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†Université Paris Panthéon-Assas, institut Léon Walras, Lemma: antoine.billot@u-paris2.fr

‡Université Paris Panthéon-Assas, institut Léon Walras, Lemma: vassili.vergopoulos@u-paris2.fr

# 1 Introduction

Suppose Alice and Bob share a common prior belief about the weather later in the day. They acquire potentially asymmetric information and update their common prior to form posterior beliefs. Furthermore, suppose their acquired information makes the exact values of their posterior beliefs for rain common knowledge—that is, both know these values, both know that they both know them, and so on ad infinitum. The seminal Agreement Theorem of Aumann [1976] establishes that, under these conditions, Alice and Bob's posterior beliefs must necessarily coincide. In other words, they cannot agree to disagree about the probability of rain.

This paper extends Aumann's Agreement Theorem in two complementary directions. First, we incorporate Knightian uncertainty or ambiguity, recognizing that agents may be unable or reluctant to assign precise probabilities to uncertain events. Instead, their beliefs may be vague, a perspective rooted in the work of Ellsberg [1961]. Following Gilboa and Schmeidler [1989] and others, we model these vague beliefs as sets of probabilities. Agents begin with an initial set of probabilities, update their beliefs upon receiving information, and form a posterior set of probabilities.

The second extension relaxes the assumption that agents observe perfectly each other's posterior beliefs. In reality, economic agents rarely have direct access to the exact values of others' beliefs; rather, they infer certain properties of these beliefs, often in a vague or informal manner. For example, Bob might infer that Alice's probability of rain is low simply because she is wearing sunglasses, while Alice might infer that Bob's probability is high because he is carrying an umbrella. Here, we assume that the properties of beliefs conveyed through information are exogenously given and may be reflected in observable actions, such as clothing choices. Within this framework, it is natural to represent a property of beliefs as a set of probability measures.

Building on these extensions, we introduce a generalized version of Aumann's Agreement Theorem. We replace the assumption of a common prior with the weaker condition that agents' prior sets of probabilities have a nonempty intersection. Instead of assuming common knowledge of exact posterior probability sets, we assume common knowledge that all of an agent's posterior probabilities satisfy a given property—meaning the agent's entire posterior set *fully* satisfies this property. Our main result establishes that agreement requires these properties to be mutually compatible. Hence, we obtain a weak form of agreement where posterior beliefs do not necessarily coincide, but their commonly known properties must nonetheless be compatible. We also derive a corollary showing that our generalized Agreement Theorem holds at every state of the world when appropriate properties are identified. However, our result applies only to specific properties of posterior sets, raising the question of whether it can be extended to broader classes of properties. To explore this, we introduce the notion of *partial* satisfaction, where only some posteriors—not necessarily all—satisfy a given property. In this direction, we establish two further results.

To illustrate the significance of our main result and its form of weak agreement, we apply them to a trade setting inspired by Milgrom and Stokey [1982]. The analysis crucially depends on the way agents react to the ambiguity they perceive, and we are lead to consider several decision-theoretic models. In some cases, the logic of weak agreement remains strong enough to preclude trade. But, in other cases, simple examples show that weak agreement remains compatible with trade.

The remainder of this paper is structured as follows. We begin by introducing the framework and notation, defining prior and posterior probabilities as well as the notion of a property. Next, we present our main result and discuss an initial set of applications. We then explore potential extensions, including alternative properties of probability sets and non-partitional information structures. Finally, we focus on applying our result to trade. All proofs are collected in the Appendix.

## 2 Framework

### 2.1 General

Let $\Omega$ be a finite state space, and let $S$ be a finite set of individuals. Each agent $i \in S$ is associated with a *partition* $\pi_i$ of $\Omega$. For every $\omega \in \Omega$, let $\pi_i(\omega)$ denote the unique cell of $\pi_i$ containing $\omega$. As usual, $\pi_i$ represents the information partition of agent $i$, meaning that at state $\omega$, agent $i$ does not know the precise state but only that it belongs to $\pi_i(\omega)$.

A *probability measure* on $\Omega$ is a function $p$ mapping the power set of $\Omega$ to the set of nonnegative real numbers, satisfying $p(\Omega) = 1$ and $p(E \cup F) = p(E) + p(F)$, for all disjoint subsets $E, F \subseteq \Omega$. Let $P(\Omega)$ denote the set of all such probability measures. For each $i \in S$, we assume a closed and convex subset $Q_i \subseteq P(\Omega)$, representing *agent i's prior beliefs* (or simply, $i$'s *prior set*). For any $E \subseteq \Omega$, define $Q_i(E) = \{q_i(E) \mid q_i \in Q_i\}$ as the set of probabilities assigned to $E$ by measures in $Q_i$.

We assume throughout the paper that a prior set of probabilities is updated according to the rule known as Full Bayes. (See Section 6 for alternative approaches.) To ensure such form of updating is well-defined, we assume $q_i[\pi_i(\omega)] > 0$, for all $i \in S$, $q_i \in Q_i$, and $\omega \in \Omega$. Under this assumption, the *Bayesian update* of $q_i$ conditional on $\pi_i(\omega)$ is given by the probability measure $q_i(\cdot \mid \pi_i(\omega))$, defined, for all $E \subseteq \Omega$, as: $q_i(E|\pi_i(\omega)) = q_i(\pi_i(\omega) \cap E)/q_i(\pi_i(\omega))$. We denote $Q_i^{\pi_i}(\omega) = Q_i[\cdot \mid \pi_i(\omega)]$ as the set of *agent i's posterior beliefs at state $\omega$* (or simply, $i$'s *posterior set* at $\omega$). Thus, each agent $i$ starts with prior beliefs $Q_i$ and, upon acquiring information at state $\omega$, updates $Q_i$ to $Q_i^{\pi_i}(\omega)$ according to Full Bayes.

### 2.2 Knowledge

Following Aumann [1976], we define, for each $i \in S$, a mapping $K_i$ from the power set of $\Omega$ to itself by setting, for every event $E \subseteq \Omega$: $K_i(E) = \{\omega \in \Omega \mid \pi_i(\omega) \subseteq E\}$. The set $K_i(E)$ consists of all states where agent $i$ knows that event $E$ holds. Thus, $K_i(E)$ is itself an event, interpreted as *"agent i knows*

$E$".

Now, consider a state $\omega \in \Omega$ and an event $E \subseteq \Omega$. The event $E$ is said to be *common knowledge at* $\omega$ if, for every finite sequence of agents $i_1, \ldots, i_n \in S$, $\omega \in K_{i_1}(K_{i_2}(\ldots K_{i_n}(E)))$. In this case, at state $\omega$, all agents know $E$, they all know that they all know $E$, and this mutual knowledge continues indefinitely.

### *2.3 Properties*

A *property* is identified with the collection of all probability measures that satisfy the given condition. Formally, such a property is defined as a subset of $P(\Omega)$.

For illustration, consider an event $E \subseteq \Omega$ and a probability level $\alpha \in [0,1]$. The property of assigning probability $\alpha$ to $E$ is represented by the set $\{p \in P(\Omega) \mid p(E) = \alpha\}$. Similarly, the property of assigning at least probability $\alpha$ to $E$ is given by $\{p \in P(\Omega) \mid p(E) \geq \alpha\}$. As another example, consider a function $\varphi : P(\Omega) \to \mathbb{R}$ and define the set $C = \{p \in P(\Omega) \mid \varphi(p) \geq \alpha\}$, for some threshold $\alpha \in \mathbb{R}$. The set $C$ thus represents the property of having a $\varphi$-attribute of at least $\alpha$. Moreover, if $\varphi$ is quasi-concave, then $C$ is convex—a property that plays a role in our results.

In this context, we say that two properties $C, C' \subseteq P(\Omega)$ are *compatible* if there exists a probability measure on $\Omega$ that satisfies both $C$ and $C'$, that is, if $C \cap C' \neq \emptyset$.

A property $C \subseteq P(\Omega)$ of probability measures can be extended to sets of probabilities in at least two distinct ways, depending on whether one applies a universal or an existential quantifier. More precisely, consider a closed and convex set $Q \subseteq P(\Omega)$. We may say that $Q$ satisfies property $C$ if *all* probability measures in $Q$ satisfy $C$. Alternatively, we may say that $Q$ satisfies property $C$ if *some* probability measure in $Q$ satisfies $C$. In our model, these two interpretations reflect varying degrees of intensity in how the posteriors satisfy a given property. Fix $\omega \in \Omega$ and, for each $i \in S$, consider a property $C_i \subseteq P(\Omega)$. We say that the posterior set $Q_i^{\pi_i}(\omega)$—agent $i$'s posteriors at $\omega$—*fully* satisfies property $C_i$ if $Q_i^{\pi_i}(\omega) \subseteq C_i$, and that $Q_i^{\pi_i}(\omega)$

*partially* satisfies property $C_i$ if $Q_i^{\pi_i}(\omega) \cap C_i \neq \emptyset$.

It is common knowledge at $\omega$ that $Q_i^{\pi_i}(\omega)$ fully satisfies property $C_i$, for all $i \in S$, if the set $\{\omega' \in \Omega \mid Q_i^{\pi_i}(\omega') \subseteq C_i\}$ is common knowledge at $\omega$, for all $i \in S$. Similarly, it is common knowledge at $\omega$ that $Q_i^{\pi_i}(\omega)$ partially satisfies property $C_i$, for all $i \in S$, if the set $\{\omega' \in \Omega \mid Q_i^{\pi_i}(\omega') \cap C_i \neq \emptyset\}$ is common knowledge at $\omega$, for all $i \in S$.

## 3 Agreement

### 3.1 Motivating example

We now present an example illustrating the various features of our approach and motivating our main result in the next section. Suppose a two-individual set $S = \{A, B\}$. The state space $\Omega$ and the partitions $\pi_A$ and $\pi_B$ are as follows: $\Omega = \{a, b, c, d, e, f\}$, $\pi_A = \{\{a, b\}, \{c, d\}, \{e\}, \{f\}\}$ and $\pi_B = \{\{b, d\}, \{a, c\}, \{e, f\}\}$. Hence, we know from the Aumann [1976] characterization of common knowledge that an event $E \subseteq \Omega$ is common knowledge at $a$ if and only if the cell in the finest common coarsening of $\pi_A$ and $\pi_B$ containing $a$ is included in $E$; That is, if $\{a, b, c, d\} \subseteq E$. Next, consider also the following properties:

$$C_A = \{p \in P(\Omega) \mid p(\{a, b, c, d\}) = 1 \text{ and } p(\{a\}) \leq 3\alpha\}$$

and

$$C_B = \{p \in P(\Omega) \mid p(\{a, b, c, d\}) = 1 \text{ and } p(\{a\}) \geq 3\beta\},$$

for some $\alpha, \beta \in [0, 1]$. Suppose next that the prior sets of probabilities are as follows:

$$Q_A = \{p \in P(\Omega) \mid p(\{a, b\}) = p(\{c, d\}) = p(\{e\}) = \frac{1}{3} \text{ and } p(\{a, d\}) \leq \alpha'\}$$

and

$$Q_B = \{p \in P(\Omega) \mid p(\{b, d\}) = p(\{a, c\}) = p(\{e, f\}) = \frac{1}{3} \text{ and } p(\{a, d\}) \geq \beta'\},$$

for some $\alpha', \beta' \in [0,1]$. Then, $A$'s posterior set is given at each of $a$ and $b$ by:

$$Q_A^{\pi_A}(a) \;=\; Q_A^{\pi_A}(b) \;=\; \{(3p, 1-3p, 0, 0, 0, 0) \mid p \in [0, \alpha']\}$$

and at each of $c$ and $d$ by:

$$Q_A^{\pi_A}(c) \;=\; Q_A^{\pi_A}(d) \;=\; \{(0, 0, 1-3p, 3p, 0, 0) \mid p \in [0, \alpha']\}.$$

As for $B$, the posterior set is given at each of $b$ and $d$ by:

$$Q_B^{\pi_B}(b) \;=\; Q_A^{\pi_B}(d) \;=\; \{(0, 1-3p, 0, 3p, 0, 0) \mid p \in [\beta', \frac{1}{3}]\}$$

and at each of $a$ and $c$ by:

$$Q_B^{\pi_B}(a) \;=\; Q_B^{\pi_B}(c) \;=\; \{(3p, 0, 1-3p, 0, 0, 0) \mid p \in [\beta', \frac{1}{3}]\}.$$

In preparation of Theorem 1 in the next subsection, we make the three following observations:

(1) $Q_A$ and $Q_B$ overlap if and only if $\alpha' \geq \beta'$,

(2) $C_A$ and $C_B$ are compatible if and only if $\alpha \geq \beta$,

(3) It is common knowledge at $a$ that $Q_A^{\pi_A}(a)$ and $Q_B^{\pi_B}(a)$ fully satisfy property $C_A$ and $C_B$ respectively if and only if $Q_A^{\pi_A}(\omega) \subseteq C_A$ and $Q_A^{\pi_B}(\omega) \subseteq C_B$ for all $\omega \in \{a, b, c, d\}$. This is in turn equivalent to $\alpha' \leq \alpha$ and $\beta \leq \beta'$.

Finally, suppose $Q_A$ and $Q_B$ overlap. Suppose also that it is common knowledge at $a$ that $Q_A^{\pi_A}(a)$ and $Q_B^{\pi_B}(a)$ fully satisfy property $C_A$ and $C_B$ respectively. It follows from the previous observations that $\alpha' \geq \beta'$, $\alpha' \leq \alpha$ and $\beta \leq \beta'$. Then, it must be that $\alpha \geq \beta$, and we obtain the compatibility of $C_A$ and $C_B$.

### *3.2 Main result*

For simplicity, we consider two-agent situations and set $S = \{A, B\}$, where $A$ stands for Alice and $B$ for Bob. (The proof of Theorem 1 makes it clear that the result continues to hold for finitely many individuals.) We now come to our main result:

**Theorem 1.** *Let $\omega \in \Omega$ and, for all $i \in S$, a convex subset $C_i \subseteq P(\Omega)$. Suppose that it is common knowledge at $\omega$ that $Q_i^{\pi_i}(\omega)$ fully satisfies property $C_i$, for all $i \in S$. If $Q_A \cap Q_B \neq \emptyset$, then $C_A$ and $C_B$ are compatible.*

Theorem 1 extends the original Agreement Theorem of Aumann [1976] by incorporating two key elements: ambiguity and imperfect observation. First, it accounts for ambiguity by representing an agent's beliefs as a set of probability measures, rather than a single, precise probability. Second, it allows for imperfect observation of others' posterior beliefs by assuming that what is common knowledge is not the exact values of these beliefs, but rather the fact that all agents' beliefs satisfy a given, agent-dependent property. Assuming the existence of at least one common prior, the theorem shows that the agents' beliefs must be mutually compatible. In doing so, it establishes a weak form of agreement in which agents may hold different posterior sets satisfying different properties, as long as these properties are compatible.

It is possible to retrieve the Agreement Theorem of Aumann [1976] from Theorem1. Indeed, suppose that $Q_A = \{q_A\}$ and $Q_B = \{q_B\}$ are singleton sets. Then, the condition $Q_A \cap Q_B \neq \emptyset$ translates into a common single prior, meaning $q_A = q_B$ in the usual sense. Moreover, consider $E \subseteq \Omega$ and $\omega \in \Omega$, and suppose, as in Aumann [1976], that the exact values of the posteriors for $E$ are common knowledge at $\omega$. Let $\alpha_A = q_A[E|\pi_A(\omega)]$ and $\alpha_B = q_B[E|\pi_B(\omega)]$ denote these commonly known values. Consider also, for all $i \in S$, the set $C_i = \{p \in P(\Omega) \mid \ p(E) = \alpha_i\}$. Since it is common knowledge at $\omega$ that the posterior of each $i \in S$ lies in $C_i$, Theorem 1 implies that $C_A \cap C_B \neq \emptyset$. Hence, we conclude that $\alpha_A = \alpha_B$. In other words, Alice and Bob must have the same posterior for $E$.

Moreover, Theorem 1 also provides a version of the Agreement Theorems obtained by Kajii and Ui [2005, 2009] and Carvajal and Correia-da-Silva [2013]. These authors extend the Agreement Theorem of Aumann [1976] to multiple priors in various ways, assuming that agents have common knowledge of the exact posterior sets of probabilities for a given event.

In this context, Kajii and Ui [2005] provide a result where the equality of *ex ante* sets implies an overlap between *ex post* sets for a given event. Kajii and Ui [2009] observe that an overlap of *ex ante* sets is sufficient for this outcome. **?** refine the results of Kajii and Ui [2005, 2009].

To better see the connection to Theorem 1, fix $E \subseteq \Omega$. For any closed and convex prior set $Q \subseteq P(\Omega)$, define $Q(E) = \{q(E) \mid q \in Q\}$, which forms a closed interval in $[0,1]$. Consider a closed interval $I_i$ in $[0,1]$, for all $i \in S$, and suppose that there exists some $\omega \in \Omega$ such that the subset $\{\omega' \in \Omega \mid Q_i^{\pi_i}(\omega')(E) = I_i\}$ is common knowledge at $\omega$. It follows that the agent $i$'s posteriors fully satisfy property $C_i$, for all $i \in S$, where $C_i = \{p \in P(\Omega) \mid p(E) \in I_i\}$. If $Q_A \cap Q_B \neq \emptyset$, Theorem 1 ensures the compatibility of $C_A$ and $C_B$. Consequently, $I_A$ and $I_B$ must overlap in the spirit of Kajii and Ui [2005, 2009] and Carvajal and Correia-da-Silva [2013].

We close with a comparison to Bach and Cabessa [2023]. In their approach, each agent has a collection of priors ordered in a lexicographic way and updates it upon information according to some adequate version of Full Bayes. Importantly, this version of Full Bayes preserves the lexicographic order. In this context, they derive in their main result a weak version of agreement where first-level posteriors agree with each other while posteriors of higher levels may disagree. This is very similar to our Theorem 1 . But we do not have a lexicographic order on the multiple priors and work with more general properties of beliefs. As a result, we cannot identify a priori which posterior lies in the intersection of properties and simply prove that a posterior exists in this intersection. (Note also that Bach and Cabessa assume the equality of the lexicographic priors while we simply assume a nontrivial overlap.)

### *3.3 State-dependent properties*

The Agreement Theorem of Aumann [1976] and its extensions under multiple priors are restricted to situations where agents have common knowledge of the exact values of their posterior beliefs regarding some event. This assumption is quite stringent and typically fails to hold in most states. In contrast, Theorem 1 is more broadly applicable, as it can be applied to all states, provided that the sets $C_A$ and $C_B$ are appropriately chosen, as detailed below.

For all $\omega \in \Omega$, let $\pi(\omega)$ denote the one element of the finest common coarsening of $\pi_A$ and $\pi_B$ containing $\omega$. Define next, for all $i \in S$:

$$C_i(\omega) := \operatorname{conv}\left\{ \bigcup_{\omega' \in \pi(\omega)} Q_i^{\pi_i}(\omega') \right\}.$$

By construction, $C_i(\omega)$ is the smallest convex property that all agents commonly know each posterior belief of agent $i$ satisfies. For example, suppose Bob has more information than Alice, meaning that $\pi_B$ is finer than $\pi_A$. In this case, we have $\pi = \pi_A$, which implies that $C_A(\omega) = Q_A^{\pi_A}(\omega)$, for all $\omega \in \Omega$. In other words, at every state, Bob knows all of Alice's posterior probabilities.

The sets $C_i(\omega)$ do not always provide enough precision to determine the exact probability of a given event, as is the case under the assumptions of Aumann [1976]. However, even when this level of precision is not attained, we can still apply Theorem 1 to derive the following corollary:

**Corollary 1.** *If $Q_A \cap Q_B \neq \emptyset$, then $C_A(\omega)$ and $C_B(\omega)$ are compatible, for all $\omega \in \Omega$.*

In fact, this corollary is equivalent to Theorem 1 in the sense that the latter can be derived from the former. Indeed, under the notation and assumptions of Theorem 1, we necessarily have $C_i(\omega) \subseteq C_i$, for all $i \in S$, by the construction of $C_i(\omega)$. Consequently, the compatibility of $C_A(\omega)$ and $C_B(\omega)$ ultimately implies the compatibility of $C_A$ and $C_B$.

The following example illustrates Corollary 1.

**Example 1.** Suppose the state space $\Omega$ and partitions $\pi_A$ and $\pi_B$ are as follows: $\Omega = \{a, b, c, d, e, f\}$, $\pi_A = \{\{a, b\}, \{c, d\}, \{e\}, \{f\}\}$ and $\pi_B = \{\{b, d\}, \{a, c\}, \{e, f\}\}$. Suppose also that the unique common prior is the uniform distribution on

$\Omega$. Then, at each state $\omega \in \{a,b,c,d\}$, we have:

$$C_A(\omega) = \text{conv}\{(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0), (0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0)\}$$

and

$$C_B(\omega) = \text{conv}\{(\frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0), (0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0)\}.$$

The unique element in $C_A(\omega) \cap C_B(\omega)$ is given by:

$$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0).$$

## 4 On extensions

### 4.1 Examples

We now examine the robustness of Theorem 1 with two examples.

**Example 2.** Consider the following state space $\Omega$ and partitions $\pi_A$ and $\pi_B$: $\Omega = \{a,b,c,d,e,f\}$, $\pi_A = \{\{a,b\},\{c,d\},\{e\},\{f\}\}$ and $\pi_B = \{\{b,d\},\{a,c\},\{e,f\}\}$. Suppose also that the two agents have the same initial set of priors $Q$ consisting of all probability measures $q$ on $\Omega$ such that $q[\pi_i(\omega)] > 0$, for all $i \in S$ and $\omega \in \Omega$. Consider the following convex subsets:

$$C_A = \{p \in P(\Omega) \mid p(\{b,c\}) = 0\} \text{ and } C_B = \{p \in P(\Omega) \mid p(\{b,c\}) = 1\}.$$

Finally, suppose that the true state is given by $\omega = b$ and set $E = \{a,b,c,d\}$. Note that $E$ is the unique cell in the finest common coarsening of $\pi_A$ and $\pi_B$ that contains $\omega$. We then know from Aumann [1976] that an event $A \subseteq \Omega$ is common knowledge at $\omega$ if and only if $E \subseteq A$. At each state in $E$, Alice and Bob have at least one posterior in $C_A$ and $C_B$ respectively. It is hence common knowledge at $\omega$ that the agent $i$'s posteriors partially satisfy property $C_i$, for all $i \in S$. Yet, in contradiction with the conclusion of Theorem 1, $C_A$ and $C_B$ are obviously disjoint.

Example 2 illustrates how Theorem 1 fails when we assume only that it is common knowledge at $\omega$ that agent $i$'s posteriors *partially* satisfy property $C_i$, for all $i \in S$. This reveals a fundamental qualitative shift in the nature of agreement introduced by ambiguity. In the case of a single prior belief, the distinction between full and partial satisfaction becomes irrelevant, as the two coincide. However, under ambiguity, the precise form of satisfaction plays a crucial role. While agreement in the sense of Aumann [1976] still holds when full satisfaction is assumed—consistent with Theorem 1—Example 2 demonstrates how partial satisfaction can result in incompatible properties.

Moreover, Geanakoplos [1989, 2021], along with Samet [1990], use possibility correspondences to model bounded rationality and errors in information processing, thereby extending the Agreement Theorem of Aumann [1976] to non-partitional structures.

To introduce possibility correspondence, note first that each information partition $\pi$ of $\Omega$ defines a function, still denoted by $\pi$, mapping each $\omega \in \Omega$ into the one cell $\pi(\omega)$ of $\pi$ containing $\omega$. This function has the following properties by construction:

(1) for all $\omega \in \Omega$, $\omega \in \pi(\omega)$,

(2) for all $\omega, \omega', \omega'' \in \Omega$, if $\omega'' \in \pi(\omega')$ and $\omega' \in \pi(\omega)$, then $\omega'' \in \pi(\omega)$,

(3) for all $\omega, \omega' \in \Omega$, $\pi(\omega) \cap \pi(\omega') \neq \emptyset$ implies $\pi(\omega) = \pi(\omega')$.

Then, a *possibility correspondence* is a function $\pi$ from $\Omega$ to the power set of $\Omega$ that merely satisfies (1) and (2).

**Example 3.** Suppose the state space $\Omega$ and partition $\pi_A$ of Alice are as follows: $\Omega = \{a, b, c, d\}$ and $\pi_A = \{\{a, b, c\}, \{d\}\}$. The possibility correspondence $\pi_B$ of Bob is given by: $\pi_B(a) = \{a, b\}$, $\pi_B(b) = \{b\}$, $\pi_B(c) = \{b, c\}$ and $\pi_B(d) = \{d\}$. Suppose that Alice and Bob share the same unique prior given by the uniform measure on $\Omega$. Consider also the following disjoint convex sets (see, Figure 1):

$$C_A = \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)\} \quad \text{and} \quad C_B = \text{conv}\{(\frac{1}{2}, \frac{1}{2}, 0, 0), (0, 1, 0, 0), (0, \frac{1}{2}, \frac{1}{2}, 0)\}.$$

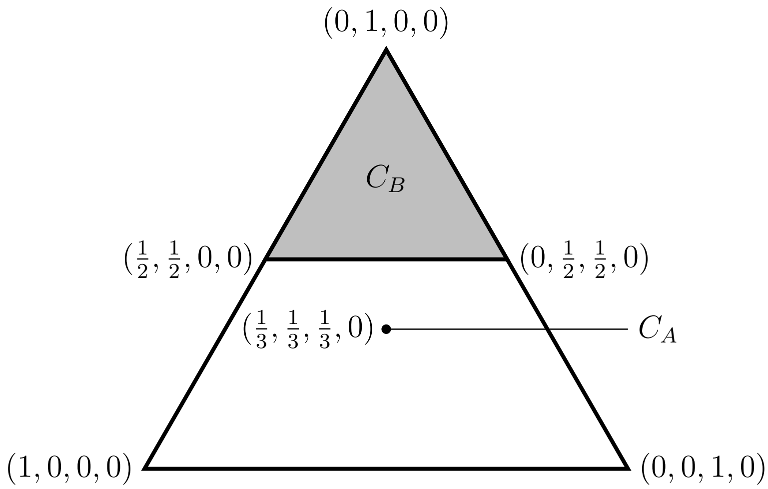


Figure 1: A failure of Theorem 1 in the nonpartitional case

A subset $E \subseteq \Omega$ is *self-evident* to every agent if, whenever $E$ holds true, every agent knows $E$, that is, if $\omega \in E$ implies $\pi_i(\omega) \subseteq E$, for all $i \in S$. Set $E = \{a, b, c\}$ and note that $E$ is self-evident to every agent. Then, by a result of Geanakoplos [1989, 2021], every subset $A \subseteq \Omega$ such that $E \subseteq A$ is commonly known at every state in $E$. It is hence common knowledge at $\omega$, for each state $\omega \in E$, that Alice's posteriors fully satisfy $C_A$ and Bob's posteriors fully satisfy $C_B$ even though $C_A$ and $C_B$ are disjoint.

Example 3 highlights a fundamental qualitative breakdown that does not stem specifically from ambiguity—since it relies on a single common prior—but rather from the inherent logic of agreement in properties versus agreement in values. While agreement in values holds in both partition-based and possibility correspondence frameworks, agreement in properties is preserved only under partitions.

### *4.2 Rectangularity*

We now extend Theorem 1 to more general classes of properties of sets of probabilities. To formalize this, we first define a property of sets of probabilities as a subset $\tilde{C}$ of $\mathcal{Q}(\Omega)$, where $\mathcal{Q}(\Omega)$ represents the collection of all convex and closed prior sets $Q \subseteq P(\Omega)$. Notably, Theorem 1 remains applicable when it is common knowledge that each agent $i$'s posterior set of probabilities satisfies property $\tilde{C}_i$, provided that each $\tilde{C}_i$ is defined as $\tilde{C}_i = \{Q \in \mathcal{Q}(\Omega) \mid Q \subseteq C_i\}$, for some property $C_i$ of probabilities. Within this framework, Propositions 1 and 2 aim to extend Theorem 1 to broader classes of properties $\tilde{C}_i$ of sets of probabilities.

Consider a closed and convex prior set $Q \subseteq P(\Omega)$ and a partition $\pi$ of $\Omega$. We say that $Q$ is *$\pi$-rectangular* if:

$$\left\{\sum_{E\in\pi} p(E)q_E,\ p \in Q \text{ and } q_E \in Q(\cdot|E), \text{ for all } E \in \pi\right\} \subseteq Q.$$

In this case, the two sets are equal. Rectangularity has been introduced in the literature in various forms by Sarin and Wakker [1998], Epstein and Schneider [2003], and Riedel [2004], and is known to characterize the property of dynamic consistency. Intuitively, the rectangularity of a prior set $Q$ with respect to a partition $\pi$ means that the posterior sets $Q(\cdot|E)$, $E \in \pi$, are independent of each other. This independence implies that the posterior sets do not provide a hedge against uncertainty.

**Proposition 1.** *Consider $\omega \in \Omega$ and a convex subset $C_i \subseteq P(\Omega)$, for all $i \in S$. Suppose that it is common knowledge at $\omega$ that $Q_i^{\pi_i}(\omega)$ partially satisfies $C_i$, for all $i \in S$. If $Q := Q_A = Q_B$ and $Q$ is $\pi_i$-rectangular, for all $i \in S$, then we have $Q^{\pi}(\omega) \cap C_i \neq \emptyset$, for all $i \in S$, where $\pi(\omega) \subseteq \Omega$ denotes the unique cell in the common finest coarsening of $\pi_A$ and $\pi_B$ that contains $\omega$.*

Theorem 1 extends the Agreement Theorem of Aumann [1976]. However, Example 1 demonstrates that this extension generally fails when full satisfaction is replaced with partial satisfaction. In this context, Proposition 1 strengthens the assumptions of Theorem 1 by introducing a common set of priors and requiring rectangularity with respect to the two information partitions. This leads to a conclusion of weak compatibility between the two properties. To understand this, note first that there always exists a prior set $Q' \subseteq P(\Omega)$ such that $Q' \cap C_i \neq \emptyset$, for all $i \in S$. Indeed, we can always set $Q' = [q_1, q_2]$ where $q_i \in C_i$, for all $i \in S$. This means that any two nonempty properties can be partially achieved by some set of probabilities. However, Proposition 1 shows that we may even assume that a subset $Q' \subseteq P(\Omega)$ such that $Q' \cap C_i \neq \emptyset$, for all $i \in S$, can be written as $Q' = Q[\cdot|\pi(\omega)]$, ie. the set of posteriors that the two agents would have held conditional on $\pi(\omega)$ achieves partially both $C_A$ and $C_B$.

### *4.3 Union-consistency*

Consider a closed and convex prior set $Q \subseteq P(\Omega)$ and a partition $\pi$ of $\Omega$. A subset $E \subseteq \Omega$ is *$\pi$-measurable* if it is a (disjoint) union of cells of $\pi$. We say that $Q$ is *union-consistent* under $\pi$ if, for all $\pi$-measurable $E \subseteq \Omega$ and all $F \subseteq \Omega$:

$$\bigcap_{\substack{G \in \pi \\ G \subseteq E}} Q(F|G) \ \subseteq \ Q(F|E).$$

To understand this definition, consider a $\pi$-measurable subset $E \subseteq \Omega$ and a subset $F \subseteq \Omega$. Then, $E$ can be written as the disjoint union $E = G_1 \cup \ldots \cup G_N$ of cells of $\pi$. Suppose $\alpha \in [0,1]$ is such that $\alpha \in Q(F|G_n)$, for all $n \in \{1, \ldots, N\}$. In other words, $\alpha$ is a possible probability value for $F$ conditional on each $G_n$. Under union-consistency, $\alpha$ must also be a possible probability value for $F$ conditional on the union of all $G_n$, ie. on $E$. For instance, every $\pi$-rectangular subset of $P(\Omega)$ is also union-consistent under $\pi$.

**Proposition 2.** *Consider $\omega \in \Omega$, $E \subseteq \Omega$ and a closed interval $I_i$ in $[0,1]$, for all $i \in S$. Suppose $\{\omega' \in \Omega \mid Q_i^{\pi_i}(\omega')(E) = I_i\}$ is common knowledge at $\omega$, for all $i \in S$. If $Q_A = Q_B$ and $Q_i$ is union-consistent under $\pi_i$, for all $i \in S$, then $I_A = I_B$.*

To the best of our knowledge, Proposition 2 offers a novel extension of the Agreement Theorem to multiple priors. We strengthen the assumptions of Kajii and Ui [2005, 2009] and Carvajal and Correia-da-Silva [2013] by assuming union-consistency and the equality of the prior probability sets. In return, we derive a stronger conclusion than the mere nontrivial overlap of $I_A$ and $I_B$, specifically showing that $I_A = I_B$.

We compare Proposition 2 to a strong version of the main result of Bach and Cabessa [2023], a paper already discussed above. In their Theorem 2., they assume a form of mutual absolute continuity and derive the exact equality of all posterior lexicographic beliefs. This is similar in spirit to Proposition 2. Like them, we assume a form of mutual absolute continuity: Indeed, given the positivity assumption of Subsection 2.1, the equality $Q := Q_A = Q_B$ ends up implying that each $q \in Q$ puts a positive probability on each cell of each of $\pi_A$ and $\pi_B$. But Proposition 2 resorts to the additional condition of union-consistency to compensate for the loss of the lexicographic structure they assume.

## 5 Application to trade

In this section, we examine the economic significance of Theorem 1 and other findings in a context of trade à la Milgrom and Stokey [1982].

Suppose $f$ a function from $\Omega$ to the reals $\mathbb{R}$ representing a *possible trade* between Alice and Bob, ie. $f(\omega)$ describes the net payment that Alice receives from Bob at state $\omega \in \Omega$. For all $i \in S$, let $x_i \in \mathbb{R}$ denote the *initial wealth* of agent $i$ and, for all $\omega \in \Omega$, assume a binary relation $\succsim_i^\omega$ on $\mathbb{R}^\Omega$ representing the preferences of agent $i$ at state $\omega$ on state-contingent monetary payoffs.

The trade $f$ is *Pareto-improving at some state* $\omega \in \Omega$ if $x_A + f \succ_A^\omega x_A$ and $x_B - f \succ_B^\omega x_B$. Then, we say that it is common knowledge at some $\omega \in \Omega$ that agents are willing to trade $f$ if the collection of states $\omega' \in \Omega$ such that $f$ is Pareto-improving at $\omega'$ is common knowledge at $\omega$. Finally, we say that trade is never Pareto-optimal if there are no state $\omega$ and function $f$ such that it is common knowledge at some $\omega$ that agents are willing to trade $f$.

### 5.1 Maxmin and unanimity preferences

We say that preferences have a *maxmin representation* if, for all $i \in S$, there exists a real-valued function $u_i$ defined on $\mathbb{R}$ and a closed and convex prior set $Q_i \subseteq P(\Omega)$ such that, for all $i \in S$, $\omega \in \Omega$ and $g, h \in \mathbb{R}^\Omega$:

$$g \succ_i^\omega h \iff \min_{q \in Q_i^{\pi_i}(\omega)} \mathbb{E}_q[u_i \circ g] > \min_{q \in Q_i^{\pi_i}(\omega)} \mathbb{E}_q[u_i \circ h].$$

We say that preferences have a *unanimity representation* if, for all $i \in S$, there exists a real-valued function $u_i$ defined on $\mathbb{R}$ and a closed and convex prior set $Q_i \subseteq P(\Omega)$ such that, for all $i \in S$, $\omega \in \Omega$ and $g, h \in \mathbb{R}^\Omega$:

$$g \succ_i^\omega h \iff \mathbb{E}_q[u_i \circ g] > \mathbb{E}_q[u_i \circ h], \text{ for all } q \in Q_i^{\pi_i}(\omega).$$

Both maxmin and unanimity representations have been given early axiomatic foundations in the decision-theoretic literature (see, for instance, Gilboa and Schmeidler [1989], Bewley [1986, 2002], and Gilboa et al. [2010]). It is also important to note that unanimity representations are inherently incomplete. As a result, an agent who is unable to rank trading against non-trading will default to maintaining the status quo, thereby refraining from trade.

Consider next the properties $C_A, C_B \subseteq P(\Omega)$ of probabilities defined by setting, for all $p \in P(\Omega)$:

$$p \in C_A \text{ iff } \mathbb{E}_p[u_A(x_A + f)] > u_A(x_A) \text{ and } p \in C_B \text{ iff } \mathbb{E}_p[u_B(x_B - f)] > u_B(x_B).$$

Suppose maxmin or unanimity representations of preferences. Clearly, the trade $f$ is Pareto-improving at some state $\omega \in \Omega$ if and only if $Q_i^{\pi_i}(\omega)$ fully satisfies property $C_i$, for all $i \in S$. Then, we also have the following: it is common knowledge at $\omega$ that agents are willing to trade $f$ if and only if it is common knowledge at $\omega$ that the agent $i$'s posteriors fully satisfy property $C_i$, for all $i \in S$.

Finally, we have:

**Proposition 3.** *Suppose $u_i$ is concave and increasing, for all $i \in S$. Then, properties $C_A$ and $C_B$ are incompatible.*

**Proposition 4.** *Suppose that the pairs $(u_i, Q_i)$ provide a maxmin or unanimity representation of preferences, for all $i \in S$. Suppose $u_i$ is concave and increasing, for all $i \in S$, and $Q_A \cap Q_B \neq \emptyset$. Then, trade is never Pareto-optimal.*

Proposition 4 leverages Theorem 1 to establish a version of the No Trade Theorem of Milgrom and Stokey [1982] within a simplified setting that nevertheless accounts for ambiguity. In doing so, it also relates to the work of Billot et al. [2000], who analyze *ex ante* trade among maxmin agents who are already fully insured and risk-averse. Their findings show that the existence of at least one prior common to all agents is sufficient to preclude trade. Proposition 4 extends this insight by demonstrating that even with the added element of asymmetric information, trade remains impossible *ex post*.

### *5.2 Maxmax and justifiable preferences*

We say that preferences have a *maxmax representation* if, for all $i \in S$, there exists a real-valued function $u_i$ defined on $\mathbb{R}$ and a closed and convex prior set $Q_i \subseteq P(\Omega)$ such that, for all $i \in S$, $\omega \in \Omega$ and $g, h \in \mathbb{R}^\Omega$:

$$g \succ_i^\omega h \iff \max_{q \in Q_i^{\pi_i}(\omega)} \mathbb{E}_q[u_i \circ g] > \max_{q \in Q_i^{\pi_i}(\omega)} \mathbb{E}_q[u_i \circ h].$$

We say that preferences have a *justifiable representation* if, for all $i \in S$, there exists a real-valued function $u_i$ defined on $\mathbb{R}$ and a closed and convex prior set $Q_i \subseteq P(\Omega)$ such that, for all $i \in S$, $\omega \in \Omega$ and $g, h \in \mathbb{R}^\Omega$:

$$g \succ_i^\omega h \iff \mathbb{E}_q[u_i \circ g] > \mathbb{E}_q[u_i \circ h], \text{ for some } q \in Q_i^{\pi_i}(\omega).$$

Maxmax representations can be obtained axiomatically by replacing the axiom of ambiguity aversion of Gilboa and Schmeidler [1989] with an axiom of ambiguity seeking. As for foundations of justifiability representations, see Lehrer and Teper [2011].

Suppose maxmax or justifiable representations. This time, the trade $f$ is

Pareto-improving in some state $\omega \in \Omega$ if and only if some probabilities in $Q_i^{\pi_i}(\omega)$ satisfy property $C_i$, for all $i \in S$. Therefore, it is common knowledge at $\omega$ that agents are willing to trade $f$ if and only if it is common knowledge at $\omega$ that $Q_i^{\pi_i}(\omega)$ partially satisfy $C_i$, for all $i \in S$. Then, we can already anticipate from Example 1 that Proposition 4 will fail in the context of maxmax or justifiable preferences.

The next example confirms this conjecture.

**Example 4.** Consider the following state space $\Omega$ and partitions $\pi_A$ and $\pi_B$: $\Omega = \{a, b, c, d, e, f\}$, $\pi_A = \{\{a, b\}, \{c, d\}, \{e\}, \{f\}\}$ and $\pi_B = \{\{b, d\}, \{a, c\}, \{e, f\}\}$. Suppose that the true state is given by $\omega = b$ and set $E = \{a, b, c, d\}$. Note that $E$ is the unique cell in the finest common coarsening of $\pi_A$ and $\pi_B$ that contains $\omega$. We then know from Aumann [1976] that an event $A \subseteq \Omega$ is common knowledge at $\omega$ if and only if $E \subseteq A$. Suppose also that the two agents have maxmax or justifiable preferences and have the same initial set of priors $Q$ consisting of all probability measures $q$ on $\Omega$ such that $q[\pi_i(\omega)] > 0$, for all $i \in S$ and $\omega \in \Omega$. Take $u_i = Id$, for all $i \in S$. In this context, consider a trade of the form $f = x - 1_{\{b,c\}}$. That is, Alice receives a sure amount of $x \in (0, 1)$ from Bob and pays him 1 only if $b$ or $c$ obtains. Now, Alice is actually willing to make this trade because she has a posterior in $Q^{\pi_A}(\omega)$ that assigns a probability of 0 to $\{b\}$. But, furthermore, Alice would also be willing to trade at each of states $c$ and $d$ because she would also have a posterior in $Q^{\pi_A}(c) = Q^{\pi_A}(d)$ assigning a probability of 0 to $\{c\}$. It is hence common knowledge that Alice wants to trade $f$. Likewise, Bob is actually willing to make the trade because he has a posterior in $Q^{\pi_B}(\omega)$ that assigns a probability of 1 to $\{b\}$. Furthermore, Bob would also be willing to trade at each of states $a$ and $c$ because he would also have a posterior in $Q^{\pi_B}(a) = Q^{\pi_B}(c)$ assigning a probability of 1 to $\{c\}$. Finally, it is here common knowledge that Alice and Bob both want to trade.

## 5.3 $\alpha$-maxmin preferences

We finally revisit Example 4 in the context of $\alpha$-maxmin preferences and show that ambiguity seeking as captured by maxmax representations is not a necessary ingredient for trade.

We say that preferences have an *$\alpha$-maxmin representation* if, for all $i \in S$, there exists $\alpha_i \in [0,1]$, a real-valued function $u_i$ defined on $\mathbb{R}$ and a closed and convex prior set $Q_i \subseteq P(\Omega)$ such that, for all $i \in S$, $\omega \in \Omega$ and $g, h \in \mathbb{R}^\Omega$:

$$g \succ_i^\omega h \Longleftrightarrow \mathbb{V}_i^\omega(g) > \mathbb{V}_i^\omega(h),$$

where, for all $k \in \mathbb{R}^\Omega$:

$$\mathbb{V}_i^\omega(k) \;=\; \alpha_i \min_{q \in Q_i^{\pi_i}(\omega)} \mathbb{E}_q[u_i \circ k] \;+\; (1-\alpha_i) \max_{q \in Q_i^{\pi_i}(\omega)} \mathbb{E}_q[u_i \circ k].$$

For the axiomatic foundations of $\alpha$-maxmin, see, for instance, Ghirardato et al. [2004], Frick et al. [2022] and Hartmann [2023]. It is common to interpret $\alpha_i$ as agent $i$'s degree of ambiguity aversion. Specifically, $\alpha_i = 1$ corresponds to full ambiguity aversion, while $\alpha_i = 0$ represents full ambiguity seeking. Intermediate values $\alpha_i \in (0,1)$ reflect more nuanced attitudes towards ambiguity. Example 4 below demonstrates that full ambiguity seeking ($\alpha_A = \alpha_B = 0$) is not a necessary condition for trade. In fact, trade can still occur when the overall ambiguity aversion remains sufficiently low, meaning that $\alpha_A + \alpha_B < 1$.

**Example 4. (cont.)** In the same context as before, we introduce $\alpha_i \in [0,1]$ and suppose that $(\alpha_i, u_i, Q)$ provides an $\alpha$-maxmin representation of the preferences of agent $i$. Suppose $\alpha_A + \alpha_B < 1$ and take any $x \in (0,1)$ such that $\alpha_A < x < 1 - \alpha_B$. Now, Alice is actually willing to make this trade at $\omega = b$ because we have: $\mathbb{V}_A^\omega(f) = \alpha_A(x-1) + (1-\alpha_A)x = x - \alpha_A > 0$. Since the same would hold true at each state in $E$, it is common knowledge at every state in $E$ that Alice wants to trade. Bob is likewise willing to make this trade at $\omega = b$ because we have: $\mathbb{V}_B^\omega(-f) = \alpha_B(-x) + (1-\alpha_B)(1-x) = 1 - \alpha_B - x > 0$. The same would hold true at each state in $E$. It is therefore common knowledge at every state in $E$ that both Alice and Bob want to trade.

## 6 Discussion

Multiple extensions of the Agreement Theorem of Aumann [1976] exist in the literature, and we refer to the surveys of Bonanno and Nehring [1997] and Ménager [2006], Ménager [2023]. To name just a few, Monderer and Samet [1989] and Bach and Cabessa [2017] study versions of the Agreement Theorem in cases where the assumption of common knowledge is replaced with weakers modalities of collective knowledge. In a similar spirit, Geanakoplos [1989, 2021] and Samet [1990] weaken the structure of individual knowledge. In contrast, Hellman [2013] weakens the assumption of a common prior and shows that common knowledge of posteriors in a case of "almost" common priors is only possible if the posteriors are "almost equal". Bach and Perea [2013] and Tsakas [2018] modify the Bayesian updating rule to include the possibility of updating on a null event. Other works call the state-space approach to uncertainty into question. For instance, Heifetz et al. [2013] study agreement in a context of unawareness while Khrennikov and Basieva [2014] and Contreras-Tejada et al. [2021] study agreement in a context of quantum uncertainty.

Of particular relevance to Section 5 are the No Trade like results à la Milgrom and Stokey [1982]. Several authors extend the Aumann conclusion of agreement into a No Trade condition which they characterize as equivalent to the assumption of a common prior. (See, Morris [1994], Samet [1998], Feinberg [2000] or Lehrer and Samet [2014].) Gizatulina and Hellman [2019] go even further by showing that a common prior is not necessary to the No Trade result as long as one commits to common knowledge of rationality.

Meanwhile, ambiguity is often rooted in the work of Keynes [1921] and Knight [1921] and receives a new impetus in the Ellsberg [1961] paradox. Schmeidler [1989] famously solves the paradox by appealing to nonadditive probabilities, also known as capacities. But sets of probabilities are also often used. For instance, see Bewley [1986, 2002], Gilboa and Schmeidler [1989], Ghirardato et al. [2004], Gilboa et al. [2010] and Lehrer and Teper [2011].

The adequate way to update a set of probabilities upon information remains a controversial issue as there are in general several equally plausible possibilities. Our choice of Full Bayes is motivated by two key reasons. First, it is well understood from an axiomatic perspective, as shown by Pires [2002] and Siniscalchi [2009]. Second, it has strong normative appeal due to its connection with dynamic consistency. In the context of maxmin preferences à la Gilboa and Schmeidler [1989], Epstein and Schneider [2003] show that Full Bayes is necessary to ensure dynamic consistency. Furthermore, in the framework of unanimity representations developed by Bewley [1986, 2002] and Gilboa et al. [2010], Full Bayes directly characterizes dynamic consistency.

Finally, Aumann's Agreement Theorem in the context of multiple priors and Full Bayes is explored by Kajii and Ui [2005, 2009], and Carvajal and Correia-da-Silva [2013]. The latter also study the alternative updating rule known as maximum likelihood. They show by means of example that agreement may fail in general and identify stronger conditions on what is commonly known to retrieve agreement.

Other authors study agreement for beliefs in the form of capacities à la Schmeidler [1989]. Zimper [2009] provides examples of disagreements due only to differences in the updating rule agents use. Dominiak and Lefort [2013] study specifically neo-additive capacities and impose the same updating rule to all agents. Under these assumptions, they derive a form of agreement. Meanwhile, Dominiak and Lefort [2013] study general capacities and obtain again agreement by supposing that all information partitions are unambiguous, a condition similar in spirit to rectangularity and union-consistency here used in Propositions 1 and 2.

# 7 Conclusion

This paper extends Aumann's Agreement Theorem in two key directions: incorporating ambiguity through multiple priors and relaxing the assumption that agents observe perfectly each other's posterior beliefs. By modeling beliefs as sets of probabilities and defining agreement in terms of commonly known properties of these sets, we establish a generalized Agreement Theorem that subsumes Aumann's original result and its extensions by Kajii and Ui [2005, 2009] and Carvajal and Correia-da-Silva [2013].

Our main result highlights that agreement hinges on the mutual compatibility of the properties that agents' posterior sets must satisfy. We further explore the implications of this theorem in economic settings, notably deriving a version of the No Trade Theorem of Milgrom and Stokey [1982] under multiple priors. Additionally, we demonstrate how trade can emerge under alternative preference structures, illustrating the broader applicability of our framework.

While our analysis focuses on fully satisfied properties of posterior sets, an important open question is whether the theorem can be extended to more general forms of partial satisfaction. Future research could also examine alternative updating rules beyond Full Bayes and explore the role of information structures that deviate from standard partition models. These directions offer promising avenues for deepening our understanding of agreement under ambiguity and its consequences in economic theory.

# Appendix

**Proof of Theorem 1:** Let $E \subseteq \Omega$ be the one cell in the common finest coarsening of $\pi_i$, $i \in S$, that contains $\omega$. Since, by assumption, it is common knowledge at $\omega$ that the agent $i$'s posteriors, ie. $Q_i^{\pi_i}(\omega)$, fully satisfy property $C_i$, for all $i \in S$, we know from Aumann [1976] that $E \subseteq \{\omega' \in \Omega \mid Q_i^{\pi_i}(\omega') \subseteq C_i\}$, for all $i \in S$. Fix $i \in S$. Then, we can find finitely many cells of $\pi_i$, denoted by $E_1, \ldots, E_N$, whose disjoint union forms all of $E$. Consider $p \in P(\Omega)$ in the intersection of all $Q_i$, $i \in S$. For all $n \in \{1,\dots,N\}$, we have $E_n \subseteq \{\omega' \in \Omega \mid Q_i^{\pi_i}(\omega') \subseteq C_i\}$. Then, take any $\omega' \in E_n$. It follows that $Q_i^{\pi_i}(\omega') \subseteq C_i$ with $p(\cdot|E_n) \in Q_i^{\pi_i}(\omega')$ by Full Bayes. Hence, we obtain $p(\cdot|E_n) \in C_i$. Furthermore, we have

$$p(\cdot|E) = \sum_{n=1}^{N} p(E_n|E) \cdot p(\cdot|E_n).$$

Since $C_i$ is convex, it follows that $p(\cdot|E)$ lies in $C_i$, and this is true for all $i \in S$. □

**Alternative proof of Theorem 1:** In this alternative proof, we suppose for simplicity that each prior $p_i$ of each agent $i \in S$ has full support. For each $i \in S$, define a function $f_i$ from the power set of $\Omega$ into $\{0,1\}$ such that, for all $E \subseteq \Omega$, we have: $f_i(E) = 1$ iff $Q_i(\cdot|E) \subseteq C_i$. Thanks to the convexity of $C_i$, we obtain that, for all $i \in S$ and disjoint $E, F \subseteq \Omega$, if $f_i(E) = f_i(F) = 1$, then $f_i(E \cup F) = 1$. In other words, $f_i$ preserves disjoint unions. The contraposition of Rubinstein-Wolinsky's (1990) Proposition 1 then yields the existence of $E \subseteq \Omega$ such that $f_i(E) = 1$ and hence $Q_i(\cdot|E) \subseteq C_i$, for all $i \in S$. Now, let $p \in Q_A \cap Q_B$. We must have $p(\cdot|E) \in Q_A(\cdot|E) \cap Q_B(\cdot|E) \subseteq C_A \cap C_B$. □

**Proof of Corollary 1:** By construction, at every state $\omega' \in \pi(\omega)$, the posterior sets $Q_A^{\pi_A}(\omega')$ and $Q_B^{\pi_B}(\omega')$ are included in $C_A(\omega)$ and $C_B(\omega)$, respectively. We know from the characterization of common knowledge of Aumann [1976] that it is common knowledge at $\omega$ that the two posterior sets $Q_A^{\pi_A}(\omega)$ and $Q_B^{\pi_B}(\omega)$ are included in $C_A(\omega)$ and $C_B(\omega)$. The result then follows from Theorem 1. □

**Proof of Proposition 1:** Set $E = \pi(\omega)$. It is sufficient to show that $Q_i(\cdot|E) \cap C_i \neq \emptyset$, for all $i \in S$, to conclude. Since it is common knowledge at $\omega$ that $Q_i^{\pi_i}(\omega)$ partially satisfy property $C_i$, for all $i \in S$, we know from the characterization of common knowledge of Aumann [1976] that $E$ is included in $\{\omega' \in \Omega \mid Q_i^{\pi}(\omega') \cap C_i \neq \emptyset\}$, for all $i \in S$. Consider now $i \in S$. We can find finitely many cells of $\pi_i$, denoted by $E_1, \dots, E_N$ whose disjoint union forms all of $E$. Then, we must have $Q_i(\cdot|E_n) \cap C_i \neq \emptyset$, for all $n \in \{1,\dots,N\}$. (Indeed, take any $\omega' \in E_n$ and observe that $Q_i(\cdot|E_n) = Q_i^{\pi}(\omega')$ by Full Bayes so that we have $Q_i(\cdot|E_n) \cap C_i = Q_i^{\pi}(\omega') \cap C_i \neq \varnothing$.) Consider then $q_n \in Q_i(\cdot|E_n) \cap C_i$, for all $n \in \{1, \ldots, N\}$. Let also $E_{N+1}, \ldots, E_m$ denote the cells of $\pi_i$ not already included in the family $\{E_1, \ldots, E_N\}$, and, for each $n \in \{N+1, \ldots, M\}$, fix an arbitrary $q_n \in Q_i(\cdot|E_n)$. Consider next: $q = \sum_{n=1}^{M} p(E_n) q_n$, where $p \in Q_i$ is arbitrary. By $\pi_i$-rectangularity, $q$ is an element of $Q_i$. It follows from Full Bayes that $q(\cdot|E)$ is an element of $Q_i(\cdot|E)$. But moreover we have: $q(\cdot|E) = \sum_{n=1}^{N} p(E_n|E) q_n$. Since $C_i$ is convex and contains $q_n$, for all $n \in \{1, \ldots, N\}$ by construction, it must also contain $q(\cdot|E)$. Hence, we obtain $Q_i(\cdot|E) \cap C_i \neq \varnothing$. □

**Proof of Proposition 2:** Let $F \subseteq \Omega$ be the one cell in the common finest coarsening of $\pi_A$ and $\pi_B$ that contains $\omega$. Since $\{\omega' \in \Omega \mid Q_i^{\pi_i}(\omega')(E) = I_i\}$ is common knowledge at $\omega$, for all $i \in S$, we know from the characterization of common knowledge of Aumann [1976] that $F \subseteq \{\omega' \in \Omega \mid Q_i^{\pi_i}(\omega')(E) = I_i\}$, for all $i \in S$. Since $Q_A = Q_B$, it is sufficient to show $Q_i(E|F) = I_i$, for all $i \in S$ to obtain $I_A = I_B$ and conclude. For $i \in S$, we can find finitely many cells of $\pi_i$, denoted by $F_1, \ldots, F_N$ whose disjoint union forms all of $F$. Suppose first $\alpha \in [0,1]$ is an element of $Q_i(E|F)$ and hence such that $\alpha = q_i(E|F)$, for some $q_i \in Q_i$ satisfying $q_i(F) > 0$. Hence, we obtain: $\alpha = \sum_{n=1}^{N} q_i(F_n|F) q_i(E|F_n)$. Now, since $F_n \subseteq F$, we must have $Q_i^{\pi_i}(\omega')(E) = I_i$, for all $\omega' \in F_n$ and $n \in \{1, \ldots, N\}$, and obtain $Q_i(E|F_n) = I_i$ and therefore $q_i(E|F_n) \in I_i$. But then $\alpha$ is an average of values all in the interval $I_i$ and hence an element of $I_i$ itself. Suppose now that $\alpha \in I_i$. For every $n \in \{1, \ldots, N\}$, we have $Q_i^{\pi_i}(\omega')(E) = I_i$, for all $\omega' \in F_n$ because $F_n \subseteq F$. Hence, we obtain $Q_i(E|F_n) = I_i$, and there exists $q_i^n \in Q_i$ such that $q_i^n(E|F_n) = \alpha$. Then, by union-consistency under $\pi_i$, there exists $q_i \in Q_i$ such that $q_i(E|F) = \alpha$. Hence, $\alpha \in Q_i(E|F)$. □

**Proof of Proposition 3:** Suppose $p \in C_A$. We then have $\mathbb{E}_p[u_A(x_A + f)] > u_A(x_A)$. By the concavity of $u_A$, we obtain $u_A[x_A + \mathbb{E}_p(f)] > u_A(x_A)$. Since $u_A$ is increasing, we further have $x_A + \mathbb{E}_p(f) > x_A$ and hence $\mathbb{E}_p(f) > 0$. Therefore, we must have $x_B > x_B - \mathbb{E}_p(f)$. Since $u_B$ is increasing, we obtain $u_B(x_B) \geq u_B[x_B - \mathbb{E}_p(f)]$. By the concavity of $u_B$, we obtain $u_B(x_B) \geq \mathbb{E}_p[u_B(x_B - f)]$. Finally, it must be the $p \notin C_B$. □

**Proof of Proposition 4:** Fix $f \in \mathbb{R}^\Omega$ and $\omega \in \Omega$. Let $C_A$ and $C_B$ be the properties as in Proposition 3. Observe that $C_A$ and $C_B$ are convex. By Proposition 3, they are also incompatible. Since $Q_A \cap Q_B \neq \emptyset$ by assumption, Theorem 1 implies that it cannot be common knowledge at $\omega$ that $Q_i^{\pi_i}(\omega)$ fully satisfy property $C_i$, for all $i \in S$. Put differently, it cannot be common knowledge at $\omega$ that agents are willing to trade $f$. Hence, trade is never Pareto-optimal. □

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